TBA equations from Y-system and discontinuity relations

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- Introduction
- O(4) nonlinear sigma model
- AdS/CFT super sigma model
- C-F-T discontinuity relations
- Summary

CFT and Integrable Models, Bologna, September 2011

O(4) nonlinear sigma model

Gromov, Kazakov, Vieira '09

Massive integrable model with $\mathrm{SU}(2) \times \mathrm{SU}(2)\ \big[\equiv \mathrm{O}(4)\big]$ symmetry

Relativistic dispersion relation: $E = \sqrt{p^2 + m^2}$

Rapidity parametrization: $E = m \cosh\theta \qquad p = m \sinh\theta$

(defrep particles)

I. S-matrix

S-matrix from: symmetries + bootstrap

$$\text{bootstrap}: \begin{cases} \text{unitarity (QM)} \\ \text{analiticity (FT)} \\ \text{crossing (FT)} \\ \text{YB} - \text{equation (integrability)} \end{cases}$$

II. Many-particle states (N particles, L: volume)

Momentum quantization modified by interaction

Weakly interacting dilute gas $(L \to \infty)$: Bethe-Yang equations

III. Thermodynamics $(N, L \to \infty)$

Mirror trick: free energy $\to$ ground state energy in finite volume

TBA integral equations:

$$\epsilon_k(\theta) + (s \star L_{k+1} + s \star L_{k-1})(\theta) = mL\delta_{k,0}\cosh\theta$$

$$L_k = \ln(1 + \mathrm{e}^{-\epsilon_k}) \qquad\qquad (k = -\infty, \dots, \infty)$$

TBA kernel function:

$$s(\theta) = \frac{g}{4\cosh\frac{\pi g\theta}{2}} \qquad\qquad \frac{1}{g} = \frac{\pi}{2}$$

Convolution:

$$(A \star B)(\theta) = \int_{-\infty}^{\infty} \mathrm{d}\theta'\, A(\theta - \theta')\, B(\theta')$$

Y-functions: $Y_k = \mathrm{e}^{-\epsilon_k}$, $L_k = \ln(1 + Y_k)$

Exponential form:

$$Y_k = \mathrm{e}^{-L\varepsilon_k}\,\mathrm{e}^{s\star(L_{k+1}+L_{k-1})}, \qquad \varepsilon_k(\theta) = \delta_{k,0}\, m \cosh\theta$$

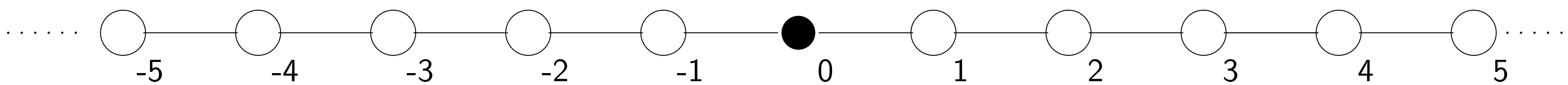
Ground state energy:

$$E_0 = -\tfrac{1}{2\pi}\sum_k \int_{-\infty}^{\infty} \mathrm{d}\theta\, \varepsilon_k(\theta) L_k(\theta) = -\tfrac{m}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta \cosh\theta L_0(\theta)$$

IV. Y-system

A. B. Zamolodchikov '91

$$Y_k^+ Y_k^- = (1 + Y_{k+1})\,(1 + Y_{k-1}), \qquad\qquad k = -\infty, \ldots, \infty$$



O(4) model TBA diagram

Meaning of massive (black) node: $\quad Y_0(\theta) \sim \mathrm{e}^{-mL\cosh\theta}$

$$f^{\pm}(\theta) = f(\theta \pm \tfrac{i}{g}), \qquad\qquad f^{[k]}(\theta) = f(\theta + \tfrac{ik}{g})$$

Route to excited state TBA equations:

– Dorey-Tateo trick

– contour deformation

– Y-system method

(Probably) all equivalent: addition of source terms

V. Excited state TBA

- same Y-system

- same asymptotic behaviour $Y_k(\theta) \sim \mathrm{e}^{-L\varepsilon_k(\theta)}$

- analytic properties (zeroes, poles) from large L asymptotic solution (Bethe Ansatz)

TBA Lemma

- $f^+ f^- = \mathcal{R}$

- roots in physical strip: $\{x_j\}_{j=1}^{\nu}$

Physical strip: $|\mathrm{Im}\,\theta| < \frac{1}{g}$

Introduce: $\tau(\theta) = \prod_{j=1}^{\nu} t(\theta - x_j)$, $\quad t(\theta) = \tanh \frac{\pi g \theta}{4}$

Solution: $\quad (t^+ t^- = 1) \quad (\tau^+ \tau^- = 1)$

$$f = \tau \, \exp\{s \star \ln \mathcal{R}\}$$

Y_k physical roots: $\{x_{k,j}\}_{j=1}^{\nu_k}$

Define: $\tau_k(\theta) = \prod_{j=1}^{\nu_k} t(\theta - x_{k,j})$

TBA integral equations:

$$Y_k = \mathrm{e}^{-L\varepsilon_k}\, \tau_k \exp\left\{s \star (L_{k+1} + L_{k-1})\right\}$$

N-particle energy:

$$E_N = \sum_{j=1}^{N} m \cosh\theta_j - \tfrac{m}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \cosh\theta L_0(\theta)$$

VI. Complete solution of the O(4) spectral problem

- $L \to \infty$: Bethe-Yang (all $1/L$ powers) $+$ Lüscher corrections (leading e^{-mL} corrections)

  (standing 1-particle states) Lüscher '86

  (general N-particle states) Bajnok, Janik '08

- $L \sim \mathcal{O}(1)$: Lattice Monte Carlo

  2d model $+$ computing power $+$ effective (cluster) algorithm $+$ simplicity of measuring mass values $\longrightarrow$ 5-digit precision MC

- $L \to 0$: Asymptotically free PT

The AdS/CFT spectral problem

SSM: superstring propagating on $AdS_5 \times S_5$ in LC gauge

Massive integrable model with $\mathrm{SU}(2|2) \times \mathrm{SU}(2|2)$ symmetry

Non-relativistic dispersion relation: $E(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$

I. S-matrix

S-matrix from symmetries + bootstrap: DONE

(also for mirror model)

Staudacher '05 Beisert '08 N. Dorey, Hofman, Maldacena '07 Arutyunov, Frolov '09

II. Many-particle states

Weakly coupled dilute gas (Bethe-Yang equations): DONE

Beisert, Staudacher '05

III. Thermodynamics

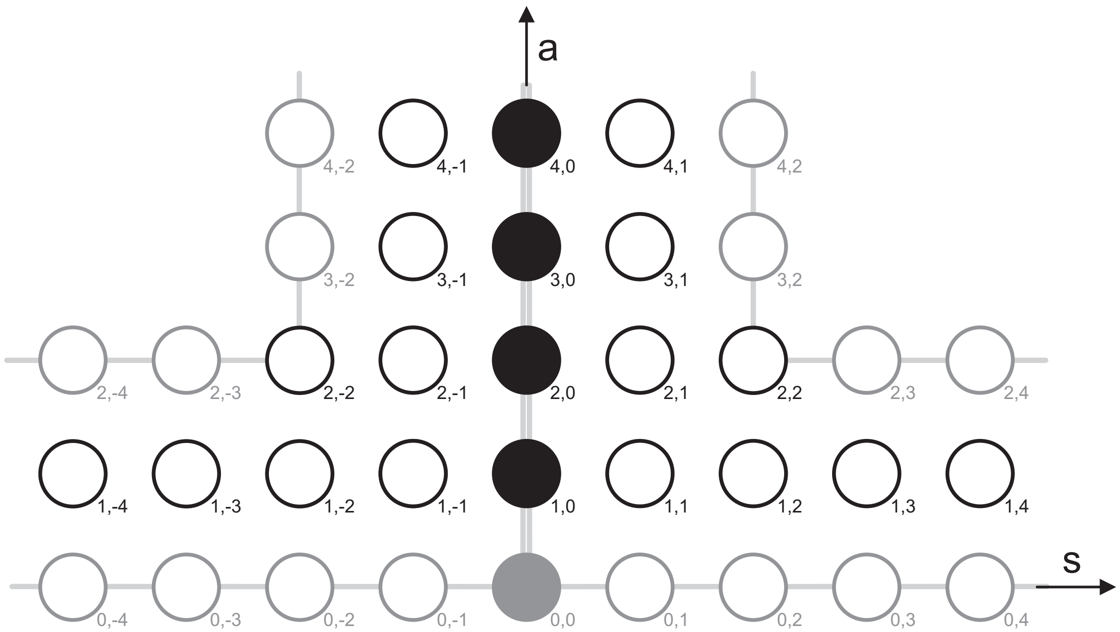
Free energy $\longrightarrow$ ground state energy in finite volume: DONE

Arutyunov, Frolov '07 Bombardelli, Fioravanti, Tateo '09 Gromov, Kazakov, Kozak, Vieira '09

Arutyunov, Frolov '09

TBA equations very complex

IV. Y-system



$AdS_5 \times S_5$ string model TBA diagram

Gromov, Kazakov, Vieira '09

Arutyunov, Frolov '09

$$Y^+_{a,s}\,Y^-_{a,s} = \frac{(1+Y_{a,s+1})\,(1+Y_{a,s-1})}{(1+1/Y_{a+1,s})\,(1+1/Y_{a-1,s})}$$

Boundary condition:

$$Y_{0,s} = \infty \qquad Y_{2,s} = \infty \quad (s \geq 3) \qquad Y_{a,2} = Y_{a,-2} = 0 \quad (a \geq 3)$$

Alternative notation: (AFS)

$$Y_Q = Y_{Q,0}$$

$$Y^{(+)}_{m|vw} = \frac{1}{Y_{m+1,1}} \qquad Y^{(-)}_{m|vw} = \frac{1}{Y_{m+1,-1}}$$

$$Y^{(+)}_{m|w} = Y_{1,m+1} \qquad Y^{(-)}_{m|w} = Y_{1,-m-1}$$

$$Y^{(+)}_{-} = -\frac{1}{Y_{1,1}} \qquad Y^{(-)}_{-} = -\frac{1}{Y_{1,-1}}$$

$$Y^{(+)}_{+} = -Y_{2,2} \qquad Y^{(-)}_{+} = -Y_{2,-2}$$

New features:

- ∞ string of bound states: $Y_Q \sim \mathrm{e}^{-L\tilde{\varepsilon}_Q}$ massive nodes

- Y-functions: not meromorphic (square root type cuts)

$$Y_{\pm}^{(\alpha)}(u): \; u + \frac{2ik}{g} \qquad k \in \mathbb{Z} \qquad |u| \geq 2$$

$$Y_m(u),\, Y_{m|vw}^{(\alpha)}(u),\, Y_{m|w}^{(\alpha)}(u): \; u \pm i\frac{m+2k}{g} \qquad k \in \mathbb{Z} \qquad |u| \geq 2$$

Real cuts: $Y_{\pm}^{(\alpha)}$

Boundary of physical strip: $Y_1, Y_{1|vw}^{(\alpha)}, Y_{1|w}^{(\alpha)}$

All other Y-s meromorphic in physical strip

ASSUMPTIONS FOR EXCITED STATES:

- same Y-system
- same large L asymptotics: $\sim \mathrm{e}^{-L\tilde{\varepsilon}_Q}$
- roots, poles from large L asymptotic (Bethe Ansatz) solution for both $\mathrm{SU}(2|2)$ wings
- square root type cuts (like for the ground state problem)

Y-system equations for w-string:

$$Y_{m|w}^{(\alpha)+}\,Y_{m|w}^{(\alpha)-} = (1+Y_{m+1|w}^{(\alpha)})\,(1+Y_{m-1|w}^{(\alpha)}) \qquad m \geq 2$$

$$Y_{1|w}^{(\alpha)+}\,Y_{1|w}^{(\alpha)-} = (1+Y_{2|w}^{(\alpha)})\,\frac{1-1/Y_{-}^{(\alpha)}}{1-1/Y_{+}^{(\alpha)}}$$

TBA equations without cuts:

$$Y_{m|w}^{(\alpha)} = \tau_{m|w}^{(\alpha)}\,\exp\left\{s\star\left(L_{m+1|w}^{(\alpha)}+L_{m-1|w}^{(\alpha)}\right)\right\} \qquad m \geq 2$$

Similar equations for: $Y_{m|vw}^{(\alpha)}$ $\;m \geq 2$ and Y_Q $\;Q \geq 2$

Modified TBA equations for: $Y_{1|w}^{(\alpha)},\ Y_{1|vw}^{(\alpha)},\ Y_1,\ Y_{\pm}^{(\alpha)}$

TBA Lemma with cuts

- $f^+ f^- = \mathcal{R}$

- roots in physical strip: $\{x_j\}_{j=1}^{\nu}$

- discontinuity:

$$\ln f(u + \tfrac{i}{g} + i\epsilon) - \ln f(u + \tfrac{i}{g} - i\epsilon) = \mathcal{V}(u) \qquad |u| \geq 2$$

Solution:

$$f = \tau \exp\{s \star \ln \mathcal{R} - s \,\check{\star}\, \mathcal{V}\}$$

Modified convolution:

$$(s \check{\star} F)(u) = \int_{|v|\geq 2} \mathrm{d}v \, s(u-v)F(v)$$

$$(s \hat{\star} F)(u) = \int_{|v|\leq 2} \mathrm{d}v \, s(u-v)F(v)$$

Missing pieces of information: discontinuities

C-F-T discontinuity (cut) relations

Cavaglia, Fioravanti, Tateo '10

Definition of extended discontinuities:

$$[f(u)]_Z = \ln f(u + \tfrac{iZ}{g} + i\epsilon) - \ln f(u + \tfrac{iZ}{g} - i\epsilon) \qquad |u| \geq 2$$

C-F-T relations:

$$\left[\ln Y_{1|vw}^{(\alpha)}\right]_{\pm 1} = \ln \frac{1-Y_-^{(\alpha)}}{1-Y_+^{(\alpha)}}$$

$$\left[\ln Y_{1|w}^{(\alpha)}\right]_{\pm 1} = \ln \frac{1-1/Y_-^{(\alpha)}}{1-1/Y_+^{(\alpha)}}$$

$$\left[\ln \frac{Y_-^{(\alpha)}}{Y_+^{(\alpha)}}\right]_{\pm 2N} = -\sum_{Q=1}^{N} [\ln(1+Y_Q)]_{\pm(2N-Q)}$$

Final relation: $\Delta = [\ln Y_1]_1$

$$[\Delta]_{\pm 2N} = \pm \sum_{\alpha} \Big\{ \Big[\ln\Big(1 - 1/Y_{\pm}^{(\alpha)}\Big)\Big]_{\pm 2N} + \sum_{m=1}^{N} \Big[\ln\Big(1 + 1/Y_{m|vw}^{(\alpha)}\Big)\Big]_{\pm(2N-m)} + \ln \frac{Y_{-}^{(\alpha)}}{Y_{+}^{(\alpha)}} \Big\}$$

$Y_{1|w}^{(\alpha)}$ TBA equation:

$$Y_{1|w}^{(\alpha)} = \tau_{1|w}^{(\alpha)} \exp\left\{ s \star L_{2|w}^{(\alpha)} + s \,\hat{\star}\, \ln \frac{1 - 1/Y_{-}^{(\alpha)}}{1 - 1/Y_{+}^{(\alpha)}} \right\}$$

Y_1 TBA equation:

$$Y_1 = \tau_1 \exp\left\{ s \star \ln\left[\frac{Y_2(1 - Y_{-}^{(+)})(1 - Y_{-}^{(-)})}{(1 + Y_2) Y_{-}^{(+)} Y_{-}^{(-)}}\right] - s \,\check{\star}\, \Delta \right\}$$

$J^{(\alpha)} = \ln \frac{Y_-^{(\alpha)}}{Y_+^{(\alpha)}}$ and Δ still in TBA equations

C-F-T relations:

– obtained from ground state TBA

– valid for asymptotic (Bethe Ansatz) solution for all states

– conjecture: C-F-T relations hold exactly for all states!

Y-system supplemented with C-F-T cut relations:

- same Y-system
- same asymptotic behaviour: $\sim \mathrm{e}^{-L\tilde{\varepsilon}_Q}$
- roots/poles from asymptotic (Bethe Ansatz) solution (large L)
- C-F-T discontinuity relations

VI. Dispersion relations

Discontinuities $J^{(\alpha)}$ and Δ from C-F-T relations

$Y_-^{(\alpha)}(u)$ and $Y_+^{(\alpha)}(u)$ analytic continuations through

real cut $|u| \geq 2$

ground state + asy (BA) $\longrightarrow$ assume for all states

Square root type cuts for $J^{(\alpha)} = \ln \frac{Y_-^{(\alpha)}}{Y_+^{(\alpha)}}$

$$J^{(\alpha)}(u + i\epsilon) = -J^{(\alpha)}(u - i\epsilon) \qquad\qquad |u| \geq 2$$

$$\frac{J^{(\alpha)}(u)}{\sqrt{4-u^2}} \qquad \text{meromorphic} \qquad |\mathrm{Im}\, u| \leq \frac{2}{g}$$

Dispersion relation:

$$\Gamma^{(\alpha)}(u) = \oint_{\Gamma_0} \mathrm{d}v\, J^{(\alpha)\prime}(v)\, K(v,u)$$

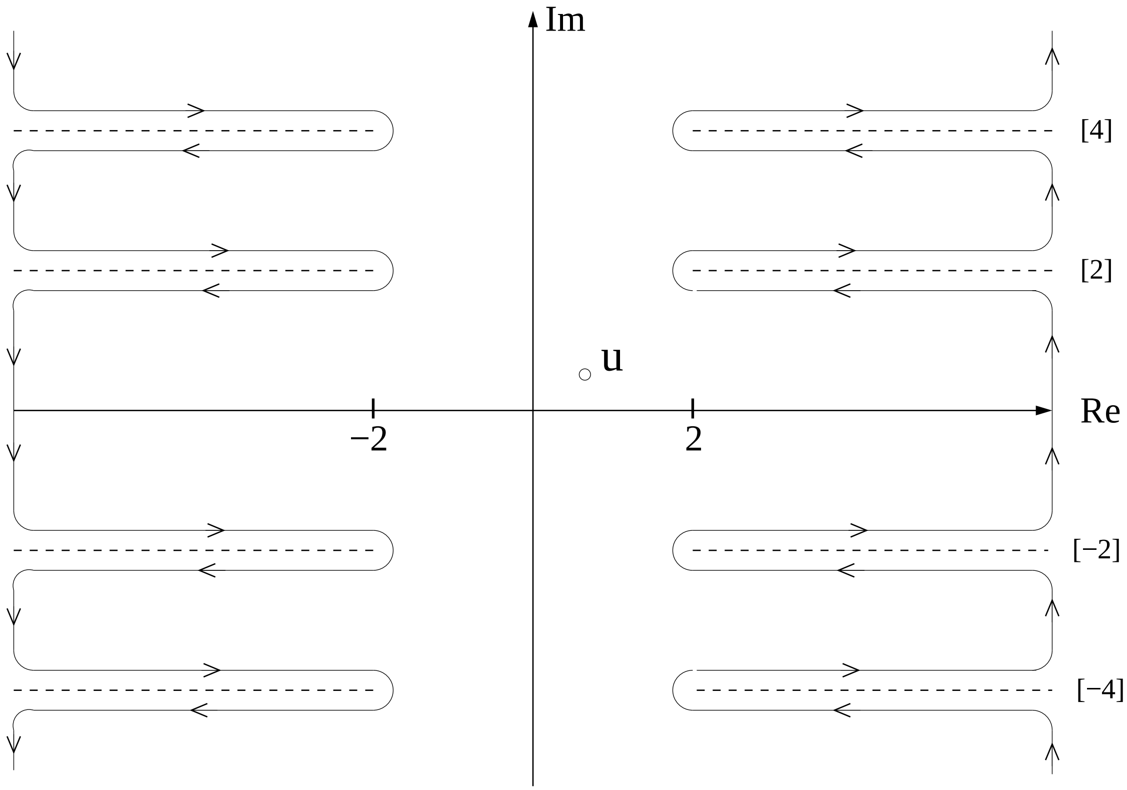
“Non-relativistic” kernel function:

$$K(v,u) = \tfrac{1}{2\pi i}\, \frac{\sqrt{4-u^2}}{\sqrt{4-v^2}(v-u)}$$

Residue theorem:

$$\Gamma^{(\alpha)}(u) = J^{(\alpha)\prime}(u) + (J^{(\alpha)\prime}\ \text{ pole terms})$$

$\Gamma^{(\alpha)}(u) =$ sum of integral of discontinuities (C-F-T)



The contour Γ_0. It goes around all positive and negative even cuts.

Result for discontinuity $J^{(\alpha)}$:

$$J^{(\alpha)}(u) = \sum_{j=1}^{N} \left\{ \ln \frac{x(u) - x_j^-}{1/x(u) - x_j^-} + \ln \frac{1/x(u) - x_j^+}{x(u) - x_j^+} \right\} + \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v \, L_Q(v) \left[K(v + \tfrac{iQ}{g}, u) - K(v - \tfrac{iQ}{g}, u) \right]$$

$x_j^{\pm}$: related to physical rapidities of the N particles

$$x(u) = (u - i\sqrt{4 - u^2})/2$$

Remarks and Summary

– Mirror TBA with Y-system method agrees with that obtained with contour deformation techniques.

Arutyunov, Frolov, Suzuki '10

– Quantization conditions: must be imposed on parameters describing roots/poles additionally (similarly to O(4) and other relativistic models).

– DRESSING: mirror TBA fully contains the contribution of the dressing phase, completely determined by the Y-system method with C-F-T relations.

$$[\Delta]_{\pm 2N} = \cdots + \cdots \pm \sum_\alpha J^{(\alpha)}$$

$$J^{(\alpha)} = j + j_L$$

$$\Delta_{\text{dress}} = \Delta_{\text{dress}}^{(\text{source})} + \Delta_{\text{dress}}^{(L)}$$

– By construction, the asymptotic (Bethe Ansatz) solution satisfies the $L_Q \to 0$ limit of the TBA equations and quantization conditions (shown previously only numerically).

- The Y-function based method of deriving excited state TBA for the AdS/CFT spectral problem, when supplemented by the C-F-T discontinuity relations leads to the correct equations.