Generalized Mathieu Equation and Liouville TBA: An unpublished work of Alexei Zamolodchikov

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CFT and Integrable Models
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- Integrable QFT in 2D
  - **Infinite volume**: (a) exact mass spectrum of asymptotic particle states. (b) factorized S-matrix, Yang-Baxter equation.
  - infinite number of "Local Integrals of Motion" (LIM): all momenta of particles remains unchanged after scattering
  - **Finite volume**: The problem is to find egenvalues and eigenstates of Local Integrals of Motion (should reproduce scattering states in the infinite volume limit)
  - Thermodynamic Bethe Ansatz (TBA), Truncated Conformal Space Approach (TCSA), "Excited states TBA"
- High Energy Limit (SG model as an example: soliton mass $M \to 0$).
  - Conformal Field Theory. Symmetry group $\Gamma = \mathsf{Vir} \otimes \overline{\mathsf{Vir}}$. Virasoro algebra Vir:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}\ , \quad c = 1 - 6(\beta - \beta^{-1})^2$$

  - Local Integral of Motion $\{I_k\}$, $k = 1, 2, \ldots, \infty$, form an (infinite-dimensional) Abelian subalgebra of Vir. The problem is to find eigenvalues and eigenstates.

- CFT analogs of Baxter commuting **T**- and **Q**-operators.
  - $\mathbf{T}(E)$ and $\mathbf{Q}(E)$ are entire functions of the spectral parameter E,

$$\mathbf{T}(E)\mathbf{Q}(E) = \mathbf{Q}(q^2\, E) + \mathbf{Q}(q^{-2}\, E), \quad q = e^{i\pi\beta^2},$$

$$Q(E_k) = 0, \quad \Rightarrow \quad \frac{Q(q^{+2}E_k)}{Q(q^{-2}E_k)} = -1; \qquad E_k \sim k^{2(1-\beta^2)}, \quad k \to \infty$$

  - Spectral zeta-function

$$Z(\nu) \sim \frac{1}{\Gamma(i\nu/2 - 1/2)} \sum_{k=1}^{\infty} E_k^{-i\nu/2(1-\beta^2)}, \qquad I_{2n-1} = Z(i(2n-1)), \quad n = 1, 2, \ldots$$

- Connection of integrable models with ODE (Voros, Dorey-Tateo, BLZ, Suzuki, Dunning, Fendley, Mangazeev, Fioravanty, Fateev, Lukyanov, S.Zamolodchikov)
  - there is one-to-one correspondence between eigenstates of LIM in the Verma module of the Virasoro algebra with $c < 1$ and potentials for 1D Schrödinger equation (rather simple potentials)
  - The zeroes E_k are the corresponding energy eigenvalues

Verma module of Vir:

$$\mathcal{V}(\Delta, c): \qquad L_n|\Delta>=0, \quad n>0; \qquad L_0|\Delta>=\Delta\,|\Delta>$$

$$|\Delta>, \quad L_{-1}|\Delta>, \quad L_{-1}^2|\Delta>, \quad L_{-2}|\Delta>, \quad \ldots$$

Differential equation for the vacuum state $|\Delta>$:

$$\Big\{-\partial_x^2+\frac{l(l+1)}{x^2}+x^{2\alpha}\Big\}\psi(x)=E\psi(x), \quad \alpha=\beta^{-2}-1, \quad \Delta=\frac{(2l+1)^2\beta^2}{16}+\frac{c-1}{24}$$

One regular and one essential singular points. Change of variables

$$\Big\{-\partial_y^2+e^{2y/\beta^2}-Ee^{2y}+(l+\tfrac{1}{2})^2\Big\}\psi(y)=0$$

For higher states there are algebraic potentials with parameters determined by the locus of stationary points for some Calogero-Moser system.

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Question: Why there is a connection of the Bethe Ansatz to ODE?

Alyosha Zamolodchikov remarked: “Here is a distinctive smell of sulphur!!!”

Alyosha Zamolodchikov at Rutgers University, 1999(?)

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Generalized Mathieu equation (GME)

Replace $\beta \to ib$ (transition from SG/CFT with $c \leq 1$ to ShG/Liouville CFT with $c \geq 25$)

$$\Big\{-\partial_y^2+P^2+e^{(\theta-y)b}+e^{(\theta+y)/b}\Big\}\psi(y)=0, \qquad \Delta=\frac{P^2}{4}+\frac{c-1}{24}$$

Two essential singular points. Becomes the (modified) Mathieu equation for self-dual point $b=1$. Assume $P, b>0$, $\theta=$ real.

$$\Big\{-\partial_y^2 + P^2 + e^{(\theta-y)b} + e^{(\theta+y)/b}\Big\}\psi(y) = 0,$$

Two uniquely defined decaying solutions

$$\begin{aligned} V_0(y) &\simeq (b/u)^{\frac{1}{2}}\, e^{-u}, \qquad u = 2b^{-1}e^{(\theta-y)b/2}, \qquad y \to -\infty, \\ U_0(y) &\simeq (bv)^{-\frac{1}{2}}\, e^{-v}, \qquad v = 2b\; e^{(\theta+y)/2b}, \qquad y \to +\infty \end{aligned}$$

Spectral problem: find the values $\{\theta_k\}$, $k = 1, 2, \ldots,$ such that

$$X(\theta_k) = 0, \qquad X(\theta) = \mathsf{Wr}[V_0, U_0], \quad \text{(spectral determinant)}$$

For these values the solution $U_0(y) \sim V_0(y)$ decays at both infinities.

Use the symmetry transformations to generate new solutions

$$\begin{aligned} \Omega_b : &\qquad y \to y + \pi i b, \qquad \theta \to \theta + i\pi b\,, \\ \Omega_{1/b} : &\qquad y \to y - \pi i/b, \qquad \theta \to \theta + i\pi/b \end{aligned}$$

$$\begin{aligned} U_1(y) &= \Omega_b\; U_0(y), \qquad & V_0(y) &= \Omega_b\; V_0(y), \\ U_0(y) &= \Omega_{1/b}\, U_0(y), \qquad & V_1(y) &= \Omega_{1/b}\, V_0(y), \end{aligned}$$

Consider all possible Wronskians among these solutions

$$\mathsf{Wr}[U_1, U_0] = -i, \quad \mathsf{Wr}[V_1, V_0] = i, \quad X(\theta) = \mathsf{Wr}[V_0, U_0], \quad X(\theta + i\pi/b) = [V_1, U_0], \ldots$$

Linear relations

$$\begin{aligned} iV_0(y) &= X(\theta + i\pi b)\, U_0(y) - X(\theta)\; U_1(y)\,, \\ iV_1(y) &= X(\theta + i\pi Q)\, U_0(y) - X(\theta + i\pi/b)\; U_1(y) \end{aligned}$$

Functional relation

$$X(\theta)\, X(\theta + i\pi Q) - X(\theta + i\pi b) X(\theta + i\pi/b) = 1, \qquad Q = b + 1/b$$

Note that $X(\theta)$ is an entire function of θ. Define

$$T(\theta) = X(\theta + i\pi b)\, X(\theta - i\pi(b - b^{-1})) - X(\theta + i\pi(b + b^{-1}))\, X(\theta - i\pi b)$$

and also $\widetilde{T}(\theta) = T(\theta)|_{b\leftrightarrow 1/b}$. It is just another connection coefficient:

$$T(\theta) U_0(y) = \Omega_b U_0(y) + \Omega_b^{-1} U_0(y) = U_1(y) + U_{-1}(y)$$

Baxter TQ-equations

$$\begin{aligned} T(\theta) X(\theta) &= X(\theta + i\pi b) + X(\theta - i\pi b), \qquad & T(\theta) = T(\theta + i\pi/b), \\ \widetilde{T}(\theta) X(\theta) &= X(\theta + i\pi/b) + X(\theta - i\pi/b), \qquad & \widetilde{T}(\theta) = \widetilde{T}(\theta + i\pi b) \end{aligned}$$

Further properties of $X(\theta)$: WKB approximation, expansions in e^{θ}, etc.

$$\frac{MR}{2}e^{\theta} = \varepsilon + \varphi * \log\left(1+e^{-\varepsilon}\right), \qquad MR = 2\pi$$

$$2\pi\varphi(\theta) = \frac{1}{\cosh\left(\theta - i\pi a\right)} + \frac{1}{\cosh\left(\theta + i\pi a\right)}, \qquad a = \frac{1-b^2}{1+b^2}$$

Asymptotic condition

$$\varepsilon(\theta) = -4PQ\theta + \text{const}, \qquad \theta \to -\infty$$

The quantity

$$X(\theta) = \exp\Big[-\frac{\pi e^{\theta}}{2\sin\left(\pi b/Q\right)} + \int \frac{\log\left(1+\exp^{-\varepsilon(\theta')}\right)}{\cosh\left(\theta-\theta'\right)}\frac{d\theta'}{2\pi}\Big]$$

satisfy the above functional equation and

$$X(\theta)\Big|_{\theta\to\infty} \sim \exp\Big(-\frac{\pi e^{\theta}}{2\sin\left(\pi b/Q\right)}\Big), \quad X(\theta)\Big|_{\theta\to-\infty} \sim \exp\left(-2PQ\theta + \text{const}\right),$$

Spectral zeta-function

$$Z(\nu) = \int_0^{\infty} dP P^{-i\nu}\frac{d}{dP}\log X(\theta, P)$$

contains all information about vacuum eigenvalues of LIM in Liouville CFT with $c \geq 25$.

$$\partial_x^2 u + [P^2 + 2e^\theta \cos x]u(x) = 0, \qquad b = 1$$

Bloch wave solutions

$$u_\pm(x) = e^{\pm\mu x} f_\pm(x), \qquad f_\pm(x) = \sum_{k=-\infty}^{\infty} \hat{f}_\pm(k) e^{ikx}$$

The exponent $\mu = \mu(\theta, P)$ is a function of P and θ. It satisfies

$$\sin^2 \pi\mu = \Delta_H(P, e^\theta)\ \sin^2 \pi P,$$

where $\Delta_H(P, e^\theta)$ is the Hill determinant

$$\Delta_H(P, e^\theta) = \det \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma_{-2} & 1 & \gamma_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \gamma_0 & 1 & \gamma_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \gamma_2 & 1 & \gamma_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad \gamma_{2n} = \frac{e^\theta}{4n^2 - P^2}.$$

Alyosha showed that

$$2\cos 2\pi\mu(\theta, P) = T(\theta, P) = \frac{X(\theta + i\pi) + X(\theta - i\pi)}{X(\theta)}$$

where $T(\theta, P)$ is the eigenvalue of the CFT analog of Baxter's **T**-operator on the Virasoro vaccum state with $\Delta = P^2/4 + 1$ and $c = 25$.

- minimal surfaces in AdS^5 with polygonal boundary (Alday, Gaiotto, Maldacena)
- Factorized basis of states in CFT (Alba, Fateev,Litvinov,Tarnopolsky)
- Generalization to massive QFT: Sine(h)-Gordon model (Lukyanov-Zamolodchikov) Simple potential $x^{2\alpha}$ is replaced by a solution of GSG equation

$$\partial_z \partial_{\overline{z}} \eta - e^{2\eta} + p(z)p(\overline{z})e^{-2\eta} = 0, \qquad p(z) = z^{2\alpha} - \mu, \quad \alpha = 1/\beta^2 - 1$$

$$\eta(z, \overline{z}) = l \log(z\overline{z}) + \eta_0 + \ldots, \qquad z, \overline{z} \to 0$$

Eigenvalues of the **Q**-operators, $Q_{\pm}(\theta)$ appear as connection coefficients for

$$[\partial_z^2 - u(z, \overline{z}) - e^{2\theta} p(z)]\psi = 0, \quad u(z, \overline{z}) = (\partial_z \eta)^2 - \partial_z^2 \eta$$

In particular, when $\beta \to 0$, $\alpha = \infty$, the polynomial

$$p(z) = \mu$$

is constant and GSG reduce the Painlevé III equation. Then eigenvalues $Q_{\pm}(\theta)$ are solutions of the Mathieu equation!

- advances in the theory of ODE (the classics did not know the Bethe Ansatz).

Is there a physical reason behind the correspondence between integrable QFT and ODE?

Alyosha Zamolodchikov at Rutgers University, 1999(?)