New "realisation" of reflection algebras and some applications.

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Plan

- Quantum algebras and "bulk" quantum integrable models
- Reflection algebras and "boundary" quantum integrable models
- New "realisation" of reflection algebras
- Conclusion

Quantum algebras and "bulk" quantum integrable models

Hopf structure and Different presentations of the quantum algebras

- Quantum algebras : $\mathcal{A} = \mathcal{Y}(g)$, $\mathcal{U}_q(\widehat{g})$,...
- Hopf structure $\{\Delta, S, \epsilon\}$: $\Delta(\mathcal{A}) \in \mathcal{A} \otimes \mathcal{A}, \quad S(\mathcal{A}) \in \mathcal{A}, \quad \epsilon(\mathcal{A}) \in \mathbb{C}$.
- quasitriangular structure $\{\exists \mathcal{R} \in \mathcal{A} \otimes \mathcal{A} \,|\, \mathcal{R}\Delta(x) = \sigma \circ \Delta(x)\mathcal{R}\}$,...

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- q-Serre-Chevalley presentation for $\mathcal{U}_q(\widehat{g})$ $\{x_i^{\pm}, q_i^{\pm h_i}\}$ (Kulish-Reshetikhin 1981, Drinfeld 1986, Jimbo 1986) :

$$q_i^{\pm h_i} q_i^{\mp h_i} = 1, \quad q_i^{h_i} q_j^{h_j} = q_j^{h_j} q_i^{h_i}, \quad q_i^{h_i}\, x_j^{\pm}\, q_i^{-h_i} = q_i^{\pm a_{ij}}\, x_j^{\pm},$$

$$[x_i^+, x_j^-] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}, \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (x_i^{\pm})^{1-a_{ij}-r}\, x_j^{\pm}\, (x_i^{\pm})^r = 0 .$$

- J presentation for $\mathcal{Y}(g)$ $\{x, J(x) \,\text{with}\, x \in g\}$, (Drinfeld 1986). For $g = sl(2)$:

$$[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f,\ [J(h), [J(e), J(f)]] = (eJ(f) - J(e)f)h$$

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- RLL presentation $\{L(u) \in \mathbb{C}^n \otimes \mathcal{A}\}$, (Faddeev-Sklyanin-Takhtadzhyan 1979) :

$$R(u, v)L_1(u)L_2(v) = L_2(v)L_1(u)R(u, v).$$

- Current presentation $\{e_i(u), f_i(u), \phi_i^{\pm}(u)\}$, (Drinfeld 1986).

Some applications of the different presentations

- q-Serre Chevalley presentation and J presentation : finite dimensional representations, construction of the R-matrices from the intertwiner equation (Jimbo 1986, Drinfeld 1986), non-local conserved current in QFT (Bernard-LeClair 1991),...

- RLL presentation : abelian algebra $l(u) = tr(L(u))$ with $[l(u), l(v)] = 0$, diagonalisation of $l(u)$ using the algebraic Bethe ansatz method (Faddeev-Sklyanin-Takhtadzhyan 1979), calculation of correlation functions (for $\widehat{g} = \widehat{sl}_2$ Lyon group's 1998),...

- Current presentation : Infinite dimensional representation (Frenkel-Jing 1988), Vertex operators and Correlation functions of infinite XXZ spin chains (Jimbo-all 1992), Scalar products of the eigenvectors of the $U_q(\widehat{sl}_3)$ spin chains (Belliard-Ragoucy-Pakuliak 2010),...

Reflection algebras and "boundary" Quantum integrable models

Coideal structure of the Reflection algebras

Consider two families of reflection algebras, $\mathcal{B}$ and $\mathcal{B}^*$, generated by the generators $\mathcal{K}(u)$ and $\mathcal{K}^*(u)$, respectively, subject to the relations (Cherednik 1984, Sklyanin 1988) :

$$\begin{aligned} R_{12}(u,v)\mathcal{K}_1(u)R_{21}(u,i(v))\mathcal{K}_2(v) &= \mathcal{K}_2(v)R_{12}(u,i(v))\mathcal{K}_1(u)R_{21}(u,v) \\ R_{12}(u,v)\mathcal{K}^*_1(u)R^{t_1}_{12}(i(u),v)\mathcal{K}^*_2(v) &= \mathcal{K}^*_2(v)R^{t_1}_{12}(i(u),v)\mathcal{K}^*_1(u)R_{12}(u,v) \end{aligned}$$

These algebras are two subalgebras of $\mathcal{A}$, let $\Phi : \mathcal{B}^{(*)} \to \mathcal{A}$:

$$\Phi(\mathcal{K}(u)) = L(u)K(u)L(i(u))^{-1}, \quad \Phi(\mathcal{K}^*(u)) = L(u)K^*(u)L(i(u))^t.$$

- **Coideal structure**, $\delta : \mathcal{B}^{(*)} \to \mathcal{A} \otimes \mathcal{B}^{(*)}$ and $\epsilon : \mathcal{B}^{(*)} \to \mathbb{C}$

$$\begin{aligned} \delta(\mathcal{K}(u)) &= L(u)\mathcal{K}(u)L(u^{-1})^{-1}, \quad \epsilon(\mathcal{K}(u)) = K(u) \\ \delta(\mathcal{K}^*(u)) &= L(u)\mathcal{K}^*(u)L(u^{-1})^{t}, \quad \epsilon(\mathcal{K}^*(u)) = K^*(u). \end{aligned}$$

remark : $\Phi = (1 \otimes \epsilon) \circ \delta$

application to "boundary" integrable models

- Abelian sub-algebra $d(u) = tr(\bar{K}(u)\mathcal{K}(u))$ with $[d(u), d(v)] = 0$, diagonnalisation of $d(u)$ using ABA (Sklyanin 1988), correlation functions (sl_2 Lyon group's 2007).
- Boundary Scattering amplitude of Integrable BQFT (Ghoshal-Zamolodchikov 1993)

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- Boundary Scattering amplitude of Integrable BQFT (Ghoshal-Zamolodchikov 1993)
- The non-local currents of the quantum Toda field theory on the half-line are given by (Mezincescu-Nepomechie 1998, Delius-MacKay 2002) :

$$A_i = c_i(x_i^+ q_i^{h_i/2} + x_i^- q_i^{h_i/2}) + w_i q_i^{h_i}.$$

Coideal structure : $\Delta(A_i) = c_i(x_i^+ q_i^{h_i/2} + x_i^- q_i^{h_i/2}) \otimes 1 + q_i^{h_i} \otimes A_i$ and $\epsilon(A_i) = w_i$

- The non-local currents of the Principal Chiral model on the half-line for $g = sl(2)$ are given by (Delius MacKay Short 2001) :

$$E = \left(J(e) - \frac{h\,e}{2}\right), \; F = \left(J(f) + \frac{h\,f}{2}\right), \; k = h + b$$

$\Rightarrow$ **On which commutation relations these generator close ?**

New "realisation" of Reflection algebras

Generalized q-Onsager algebras

The generalized $q-$Onsager algebra $\mathcal{O}_q(\widehat{g})$ is an associative algebra with unit 1, elements A_i and scalars $\rho_{ij}^k \in \mathbb{C}$ with $i,j \in \{0,1,...,n\}$, $k \in \{0,1,...,[-\frac{a_{ij}}{2}]-1\}$ and $l \in \{0,1,...,-a_{ij}-1-2\,k\}$ (Baseilhac-Belliard 2010) :

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} \mathsf{A}_i^{1-a_{ij}-r} \mathsf{A}_j \, \mathsf{A}_i^r =$$

$$\sum_{k=0}^{[-\frac{a_{ij}}{2}]-1} \rho_{ij}^k \sum_{l=0}^{-2\,k-a_{ij}-1} \gamma_{ij}^{kl} \, \mathsf{A}_i^{-2\,k-a_{ij}-1-l} \mathsf{A}_j \, \mathsf{A}_i^l \ ,$$

- **Coideal structure**, $\delta : \mathcal{O}_q(\widehat{g}) \to \mathcal{U}_q(\widehat{g}) \otimes \mathcal{O}_q(\widehat{g})$.

$$\delta(A_i) = c_i(x_i^+ q_i^{h_i/2} + x_i^- q_i^{h_i/2}) \otimes 1 + q_i^{h_i} \otimes A_i$$

with relations between the c_i and the ρ_{ij}^k.

$$\epsilon(A_i) = a_i \tag{3.1}$$

with constraints on the a_i.

Twisted Yangian case $\mathcal{Y}^+(sl_2)$

- Let $\mathcal{Y}^+(sl_2)$ denote the associative unital algebra with three generators $\{k, E, F\}$ and the defining relations :

$$[k, E] = 2\,E, \qquad [k, F] = -2\,F,$$
$$\Big[E, [E, [E, F]]\Big] = -12\,E\,k\,E, \qquad \Big[F, [F, [F, E]]\Big] = 12\,F\,k\,F$$

with Coideal structure $\delta : \mathcal{Y}^+(sl_2) \to \mathcal{Y}(sl_2) \otimes \mathcal{Y}^+(sl_2)$ and $\epsilon : \mathcal{Y}^+(sl_2) \to \mathbb{C}$

$$\begin{aligned}
\delta(k) &= h \otimes 1 + 1 \otimes k \\
\delta(E) &= \left(J(e) - \frac{h\,e}{2}\right) \otimes 1 + 1 \otimes E - e \otimes k \\
\delta(F) &= \left(J(f) + \frac{h\,f}{2}\right) \otimes 1 + 1 \otimes F + f \otimes k \\
\epsilon(k) &= b, \quad \epsilon(E) = \epsilon(F) = 0.
\end{aligned}$$

Some applications

- $\mathcal{O}_q(\widehat{g})$ is the symmetry algebra of the affine Toda field theories on the half-line with non preserving boundary conditions has been obtained and allowed to classify admissible boundary condition (scalar $\propto \epsilon(A_i)$, dynamic) : $O_q(\widehat{g})$. (Baseilhac and Belliard 2010). For the scalar admissible boundary condition we recover the known results (Corrigan-all 1994 1995).

- Dynamical admissible boundary condition are given by realisation of $\mathcal{O}_q(\widehat{g})$. As exemples, or the case $\widehat{g} = a_1^{(1)}, a_2^{(2)}, a_1^{(2)}$ realisation by finite dimensional algebra is given by special cases of the Zhedanov or Askey-Willson algebra (Zhedanov 1992).
One can obtain "Dynamical" K matrices from intertwiner equation, with $\psi : \mathcal{O}_q(\widehat{g}) \rightarrow \mathcal{AW}$ (Baseilhac-Koizomi 2004, Belliard-Fomin 2011) :

$$K^d(u)\left(\pi_u \otimes \psi\right) \circ \delta(\mathsf{A}_i) = \left(\pi_{u^{-1}} \otimes \psi\right) \circ \delta(\mathsf{A}_i)\, K^d(u).$$

remark : The scalar K matrices are obtained for $\psi \rightarrow \epsilon$ (Nepomechie 2002, Delius MacKay 2002,...)

Conclusion

- Dynamical amplitudes for quantum Toda fields theories related to $O_q(a_2^{(2)})$ and $O_q(a_2^{(1)})$, weak strong duality ? (Baseilhac-Belliard in preparation).

- Same result could be obtain for the Principal Chiral model : $\mathcal{Y}^t(a_1), \mathcal{Y}^t(a_2)$ (Belliard-Crampé, in preparation).

- For the case $O_q(a_1^{(1)})$ a systematic derivation of the integrable hierarchy has been obtained. (Baseilhac-Shigechi 2009, Baseilhac-Belliard 2010)

- For the case $\mathcal{O}_q(\widehat{sl}_2)$, the q-Onsager algebra, one can show that is the symmetry algebra of the half XXZ spin chain with generic boundary condition and for diagonal boundary conditions the symmetry is given by the augmented q-Onsager algebra (Ito-Terwilliger 2009). (Baseilhac 2004, Baseilhac-Belliard in preparation).