TBA and double-wrapping corrections for the ground-state of the γ_i-deformed AdS/CFT

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based on:

- C. Ahn, Z. Bajnok, D. B., R. I. Nepomechie; "TBA, NLO Luscher correction, and double wrapping in twisted AdS/CFT"; arXiv:1108.4914 [hep-th]

Outline

Introduction

conformal N=4 SUSY Yang-Mills $\Leftrightarrow$ IIB superstring on $AdS_5 \times S^5$ [Maldacena '97; Gubser, Klebanov, Polyakov ; Witten '98]

$$g^2 = \frac{\lambda_{AdS_5}}{16\pi^2} = \frac{R^4_{AdS_5}}{16\pi^2\alpha'^2}$$

$$\boxed{\Delta(\lambda, N) = E(\lambda, g_s)}$$

where $\mathfrak{D}\,\mathcal{O} = \Delta\,\mathcal{O}$ such that $\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle \propto \frac{1}{|x-y|^{2\Delta}}$

local gauge invariant operators $\mathcal{O} = Tr(\psi^A_\alpha D_\mu \phi_a D_\nu \bar{\psi}^{\bar{A}}_{\dot{\alpha}})$

- **strong/weak** coupling dualities $\Rightarrow$ hard to test
- breakthrough: discovery of **integrability** on both sides of the correspondences in the planar limit ($g_s \propto 1/N \to 0$)
- gauge operators $\leftrightarrow$ integrable spin chains [Minahan, Zarembo '02;'08]
- string solutions $\leftrightarrow$ integrable non-linear sigma models [Bena, Polchinski, Roiban '03]

Outline

Introduction

Integrability in AdS/CFT

Integrability in deformed AdS/CFT

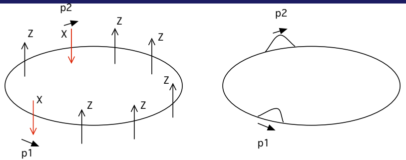
Vacuum Lüscher-like corrections

TBA for deformed AdS/CFT

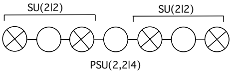
Conclusions

The AdS/CFT S-matrix

in **AdS**/**CFT**: asymptotic states are excitations on the vacuum $tr\,(Z^L)$ at large volume L



$$PSU(2,2|4) \to PSU(2|2) \times PSU(2|2) \ltimes \mathbb{R}^3$$



$\Rightarrow$ the invariance under this superalgebra determines the fundamental S-matrix elements up to the scalar factor:

$$\left[\mathfrak{J}_1 + \mathfrak{J}_2, S_{12\,SU(2|2)}(p_1, p_2)\right] = 0 \Rightarrow S(p_1, p_2) = S_0\left[S_{SU(2|2)} \otimes S_{SU(2|2)}\right](p_1, p_2)$$

- overall scalar factor S_0 contains **dressing factor**

  $$S_0(p_k, p_l) = \frac{x_k^- - x_l^+}{x_k^+ - x_l^-}\,\frac{1 - 1/x_k^+ x_l^-}{1 - 1/x_k^- x_l^+}\,\sigma^2(x_k, x_l)\,, \text{ where } x^{\pm}(u) = \frac{1}{2}\left[u \pm i/2 + \sqrt{u \pm i/2 - 4g^2}\right]$$

  conjectured for agreement between weak and strong coupling spectrum [Arutyunov, Frolov, Staudacher '04; Beisert, Hernandez, Lopez '06]
- fixed by the **crossing symmetry** [Janik '06; Volin '09]
- bound-state S-matrices $S^{Q_1 Q_2}(p_1, p_2)$ determined by invariance under Yangian symmetry $Y(su(2|2))$ [Arutyunov, Frolov '08; Arutyunov, de Leeuw, Torrielli '09]

The AdS/CFT Bethe Ansatz equations

- **all-loop asymptotic** BAEs for the full $PSU(2,2|4)$ superalgebra of SYM [Beisert, Staudacher '05]

$$
\begin{aligned}
e^{ip_k L} &= \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{I}}} \left[S_0(p_k,p_l) \left(\frac{x_k^+ - x_l^-}{x_k^- - x_l^+} \sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}} \right)^{\frac{1+\eta}{2}} \right]^2 \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \left(\frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \right)^{\eta} \\
1 &= \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}} \\
1 &= \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}
\end{aligned}
$$

- $\eta = \pm 1$ depending on grading $SU(2)/SL(2)$
- **non-diagonal** scattering $\Rightarrow$ auxiliary variables describing particles' inner degrees of freedom
- BAEs can be obtained by imposing momenta quantization conditions and analyticity conditions on the **transfer matrix** eigenvalues

$$
e^{ip_k L} T(p_k;\{p_l\}) = 1\,; \quad T(p_k;\{p_l\}) = \sum_b (-1)^{F_b} S_{c_1 k}^{c_2 b}(p_k,p_1) S_{c_2 k}^{c_3 b}(p_k,p_2)\dots S_{c_N k}^{c_1 b}(p_k,p_N)
$$

Wrapping corrections

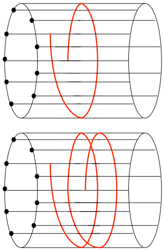
- **SU(2) sector**: all-loop **asymptotic** BAEs for long range spin chain models [Beisert, Dippel, Staudacher '04] with **dressing factor** [Beisert, Eden, Staudacher '06]

$$\left[\frac{x^+(u_j)}{x^-(u_j)}\right]^L = \sigma(u_k, u_j; g)^2 \prod_{\substack{k=1 \\ k\neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$

$$H = H_{Heis} + g^2 H_{next-to-nearest} + g^4 H_{next-to-next-to-nearest} + \cdots$$
$$\Delta = L + g^2 E_{Heis} + g^4 E_2 + g^6 E_3 + \ldots$$

Beisert-Staudacher BAEs are **asymptotic**: affected by **wrapping corrections** when interaction range (loop order) exceeds L: at order g^{2L}

double wrapping corrections when interaction range wraps twice the chain: at order g^{4L}



Outline

Integrability in deformed AdS/CFT

- marginal deformations of $\mathcal{N}=4$ SYM $\Rightarrow \mathcal{N}=1$ [Leigh, Strassler '95]

$$\mathcal{W}_{\mathcal{N}=4} = gTr(X[Y,Z]) \Rightarrow \mathcal{W}_{\mathcal{N}=1} = gTr\left[X(YZ - qZY) + \frac{h}{3}(X^3 + Y^3 + Z^3)\right]$$

- β-deformation: $h=0$ and $q = e^{2i\pi\beta}$ (gravity dual by [Lunin, Maldacena '05])

$$\mathcal{W}_\beta = gTr\left[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_2\Phi_3\right]$$

- non-supersymmetric generalization: $\gamma_1, \gamma_2, \gamma_3$ deformation

$$\mathcal{W}_{\gamma_i} = Tr\sum_{n>m=1}^{3}|e^{i\pi\epsilon_{mni}\gamma_i}\Phi_m\Phi_n - \Phi_n\Phi_m e^{-i\pi\epsilon_{mni}\gamma_i}|^2 + Tr\sum_{m=1}^{3}[\Phi_m, \bar{\Phi}_m]^2$$

  of isometric angles in S^5: $\phi_i(2\pi) - \phi_i(0) = 2\pi(n_i - \epsilon_{ijk}\gamma_j J_k)$
- classical integrability: Lax pair for real and special complex parameters [Frolov '05]
- gauge: integrable **twisted** spin chain model in SU(2) and SU(3) 1-loop [Roiban '03; Berenstein, Cherkis '04] and SU(2) higher loops [Frolov, Roiban, Tseytlin '05]

$$\left[\frac{x^+(u_j)}{x^-(u_j)}\right]^L e^{-2i\pi\beta L} = \sigma(u_k, u_j; g)^2 \prod_{\substack{k=1\\k\neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$

- Beisert and Roiban conjectured all-loop BAEs for all the sectors [Beisert, Roiban '05]

Twisted S-matrix and BAE [Ahn, Bajnok, D.B., Nepomechie '10]

- **Reshetikhin twist** of the AdS/CFT S-matrix: $\tilde{S}_{12}(p_1,p_2)=F_{12}S_{12}(p_1,p_2)F_{12}$

with $F_{12}=e^{i\gamma_1(h\otimes\mathbb{I}\otimes\mathbb{I}\otimes h-\mathbb{I}\otimes h\otimes h\otimes\mathbb{I})}$; $h=\mathrm{diag}(1/2,-1/2,0,0)$ on $(1,2,3,4)\otimes(\dot{1},\dot{2},\dot{3},\dot{4})$

- **+ twisted boundary conditions** $\Rightarrow$ AdS/CFT twisted transfer matrix

$$\tilde{t}(u)=Str_{a\dot{a}}M_{a\dot{a}}\tilde{S}_{a\dot{a}1\dot{1}}(u,p_1)\ldots\tilde{S}_{a\dot{a}N\dot{N}}(u,p_N)\,;\;\; M=F_{13}^2F_{23}^2=e^{i(\gamma_3-\gamma_2)hJ}\otimes e^{i(\gamma_3+\gamma_2)hJ}$$

- resulting BAEs match **Beisert-Roiban's BAEs**

$$e^{ip_kL}=\prod_{\substack{l=1\\l\neq k}}^{K^{\mathrm{I}}}\left[S_0(p_k,p_l)\left(\frac{x_k^+-x_l^-}{x_k^--x_l^+}\sqrt{\frac{x_l^+x_k^-}{x_l^-x_k^+}}\right)^{\frac{1+\eta}{2}}\right]^2\prod_{\alpha=1}^{2}(c_\alpha)^{\frac{1+\eta}{2}}\prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}}\left(\frac{x_k^--y_l^{(\alpha)}}{x_k^+-y_l^{(\alpha)}}\sqrt{\frac{x_k^+}{x_k^-}}\right)^{\eta}$$

$$c_\alpha=\prod_{l=1}^{K^{\mathrm{I}}}\frac{y_k^{(\alpha)}-x_l^+}{y_k^{(\alpha)}-x_l^-}\sqrt{\frac{x_l^-}{x_l^+}}\prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}}\frac{v_k^{(\alpha)}-w_l^{(\alpha)}+\frac{i}{g}}{v_k^{(\alpha)}-w_l^{(\alpha)}-\frac{i}{g}}$$

$$(c_\alpha)^{1-\eta}=\prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}}\frac{w_k^{(\alpha)}-v_l^{(\alpha)}-\frac{i}{g}}{w_k^{(\alpha)}-v_l^{(\alpha)}+\frac{i}{g}}\prod_{\substack{l=1\\l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}}\frac{w_k^{(\alpha)}-w_l^{(\alpha)}+\frac{2i}{g}}{w_k^{(\alpha)}-w_l^{(\alpha)}-\frac{2i}{g}}$$

$$c_1=e^{i\gamma_1N/2}e^{-i\gamma_1\dot{m}_1/2}e^{i(\gamma_3-\gamma_2)J/2}\,,\quad c_2=e^{-i\gamma_1N/2}e^{i\gamma_1m_1/2}e^{i(\gamma_3+\gamma_2)J/2}$$

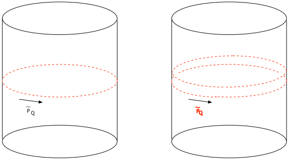
- and *operational* **twist** transfer matrix [Arutyunov, de Leeuw, van Tongeren '10]
- and asymptotic solution of β**-deformed Y-system** [Gromov, Levkovich-Maslyuk '10]

also in the $sl(2)$ grading $\eta=-1$: $\tilde{c}_1=e^{i\gamma_1(\dot{\tilde{m}}_1-2m_2)/2}e^{i(\gamma_3-\gamma_2)J/2}$, $\tilde{c}_2=e^{-i\gamma_1(\tilde{m}_1-2m_2)/2}e^{i(\gamma_3+\gamma_2)J/2}$

Outline

Twisted vacuum Lüscher-like corrections (LO & LO^2)

we derived the leading and next-to-leading **Lüscher-like corrections** for the vacuum of a relativistic theory with **twisted boundary conditions**



$$E_0^d(L) = -\lim_{R\to\infty}\frac{1}{R}\ln\tilde{Z}^d(L,R) = -\lim_{R\to\infty}\frac{1}{R}\ln \mathrm{Tr}(e^{-H(R)L}e^{i\gamma h})$$

that in the **deformed** AdS/CFT case corresponds to

- **single wrapping**

$$E_0^d = -\mathrm{Tr}(e^{i\gamma h})m\int\frac{d\theta}{2\pi}\cosh\theta\, e^{-Lm\cosh\theta}$$

$$\Updownarrow$$

$$-\sum_{Q=1}^{\infty}\mathrm{Str}_Q(e^{2i\gamma_l h}\otimes e^{2i\gamma_r h})\int\frac{d\tilde{p}}{2\pi}e^{-L\tilde{E}_Q(\tilde{p})} = -(2-[2]_q)(2-[2]_{\dot{q}})\sum_Q Q^2\int\frac{d\tilde{p}}{2\pi}e^{-L\tilde{E}_Q(\tilde{p})}$$

- **double wrapping**: quadratic term

$$+\mathrm{Tr}(e^{i\gamma h})^2\frac{m}{2}\int\frac{d\theta}{2\pi}\cosh\theta\, e^{-2Lm\cosh\theta}$$

$$\Updownarrow$$

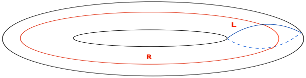
$$+\frac{1}{2}\sum_{Q=1}^{\infty}\mathrm{Str}_Q(e^{2i\gamma_l h}\otimes e^{2i\gamma_r h})^2\int\frac{d\tilde{p}}{2\pi}e^{-2L\tilde{E}_Q(\tilde{p})} = \frac{1}{2}(2-[2]_q)^2(2-[2]_{\dot{q}})^2\sum_Q Q^4\int\frac{d\tilde{p}}{2\pi}e^{-2L\tilde{E}_Q(\tilde{p})}$$

where $q = e^{i\pi\gamma_l}$; $\dot{q} = e^{i\pi\gamma_r}$; $\gamma_l = (\gamma_3-\gamma_2)L/2$; $\gamma_r = (\gamma_3+\gamma_2)L/2$

$$[N]_q = \frac{q^N-q^{-N}}{q-q^{-1}} = q^{N-1}+q^{N-3}+\ldots+q^{3-N}+q^{1-N}$$

The mirror theory

(1+1)D **relativistic** QFT on a torus $\Rightarrow$ invariance under exchange of space and time [Zamolodchikov '90] $(H(L) = \tilde{H}(R))$



$$Z(R, L) = Tr\, e^{-RH(L)} \underset{R\to\infty}{\approx} e^{-RE_0(L)}$$

$$\tilde{Z}(R, L) = Tr\, e^{-L\tilde{H}(R)} \underset{R\to\infty}{\approx} e^{-RLf(L)} \quad \text{with } T = 1/L$$

- **non-relativistic** case $(H(L) \neq \tilde{H}(R))$: the double Wick rotation defines two **different theories** (physical and mirror) related by analytic continuation

$$\boxed{\begin{aligned} E(p) = \sqrt{1 + 4g^2 \sin^2(p/2)} \quad &\to \quad i\tilde{p} \\ p \quad &\to \quad i\tilde{E}(\tilde{p}) = 2\, i\, \text{arcsinh}\left(\sqrt{1 + \tilde{p}^2}/2g\right) \end{aligned}}$$

- by analytically continuing the AdS/CFT S-matrix $\Rightarrow$ **mirror S-matrix**

$$\tilde{S}(\tilde{p}_1, \tilde{p}_2) = S(p_1, p_2)$$

$\Rightarrow$ **mirror asymptotic BAEs** [Arutyunov, Frolov '07]:

- defined in the **mirror** kinematic region: $|x^{\pm}_{ph}(u)| > 1 \to \mathcal{I}m[x^{\pm}_{mir}(u)] < 0$

$$x_{mir}(u) = \frac{1}{2}(u - i\sqrt{4 - u^2})\,; \quad \tilde{p} = gx_{mir}(u - i/g) - gx_{mir}(u + i/g) + i$$

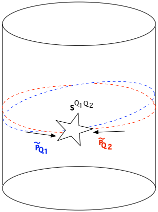
Twisted vacuum Lüscher-like corrections (NLO)

term involving **bound-states S-matrix**

$$+m\int \frac{d\theta_1}{2\pi}\cosh\theta_1\, e^{-Lm\cosh\theta_1}\int\frac{d\theta_2}{2\pi}e^{-Lm\cosh\theta_2}(i\partial_{\theta_1})\mathrm{Tr}\left(e^{i\gamma h}\ln S(\theta_1-\theta_2)\right)$$

$$\Updownarrow$$

$$+\sum_{Q_1,Q_2}\int\frac{d\tilde{p}_1}{2\pi}e^{-L\tilde{E}_{Q_1}(\tilde{p}_1)}\int\frac{d\tilde{p}_2}{2\pi}e^{-L\tilde{E}_{Q_2}(\tilde{p}_2)}i\partial_{\tilde{p}_1}\mathrm{STr}_{Q_1Q_2}\left(e^{2i\gamma_l h}\otimes e^{2i\gamma_r h}\ln S^{Q_1Q_2}(\tilde{p}_1,\tilde{p}_2)\right)$$



simultaneous diagonalization of S-matrix and twist-matrix

$$U^{-1}SU=\Lambda\,,\quad U^{-1}e^{2i\gamma_l h}U=A\,,\quad U^{-1}e^{2i\gamma_r h}U=\dot{A}$$

$$\Rightarrow \mathrm{STr}(e^{2i\gamma_l h}\otimes e^{2i\gamma_r h}\log\mathcal{S})=\mathrm{STr}(\dot{A})\,\mathrm{STr}(A)\log S^{Q_1Q_2}_{sl(2)}+\sum_i(-1)^{F_i}\left(\mathrm{STr}(A)\dot{A}_i+\mathrm{STr}(\dot{A})A_i\right)\log\Lambda_i$$

NLO energy correction

$$E_0^{(2,2)}=\sum_{Q_1,Q_2}Q_1Q_2\int\frac{d\tilde{p}_1}{2\pi}e^{-L\tilde{\epsilon}_{Q_1}(\tilde{p}_1)}\int\frac{d\tilde{p}_2}{2\pi}e^{-L\tilde{\epsilon}_{Q_2}(\tilde{p}_2)}i\partial_{\tilde{p}_1}\Big[Q_1Q_2(2-[2]_q)^2(2-[2]_{\dot{q}})^2\log S^{Q_1Q_2}_{sl(2)}(\tilde{p}_1,\tilde{p}_2)$$
$$+\sum_i(-1)^{F_i}\left(\dot{A}_i(2-[2]_q)^2+A_i(2-[2]_{\dot{q}})^2\right)\log\Lambda_i^{Q_1Q_2}(\tilde{p}_1,\tilde{p}_2)\Big]$$

we need to to calculate the **determinant of the bound-state S-matrix**

S-matrix determinants and result

we use the $su(2)_L \otimes su(2)_R$ decomposition (L, R rotation generators of $su(2|2)$) of the $su(2|2)$ **bound-states S-matrix** derived in [Arutyunov, de Leeuw, Torrielli '09] and of the twisting matrix

$$\sum_i (-1)^{F_i} A_i \log \Lambda_i^{Q_1,Q_2} = \sum_{(s_L,s_R)} \mathrm{STr}[(\mathbb{I}\otimes e^{i\gamma_R h})\log S^{Q_1 Q_2}(s_L,s_R)]$$
$$= \sum_{(s_L,s_R)} (-1)^{s_R}(2s_L+1)\,[2s_R+1]_q \log\det S^{Q_1 Q_2}(s_L,s_R)$$

and calculate $\det S^{Q_1 Q_2}(s_L,s_R)$ for some $Q_1, Q_2 \Rightarrow$ conjecture of general expression in terms of **few simple building blocks**

$$U_0 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,,\quad U_1 = \sqrt{\frac{x_1^+}{x_1^-}}\,,\quad U_2 = \sqrt{\frac{x_2^-}{x_2^+}}\,,\quad U_3 = \frac{x_1^+ x_2^+ - 1}{x_1^- x_2^- - 1}\,,\quad S_Q = \frac{u_1 - u_2 - \frac{iQ}{g}}{u_1 - u_2 + \frac{iQ}{g}}$$

then the final result is ($\mathcal{K}^{Q_1 Q_2} = \sum_{j=0}^{Q_1-1}(Q_2 - Q_1 + 2j + 1)\sum_{k=1}^{Q_1-j-1}\log S_{Q_2-Q_1+2j+2k}$)

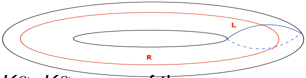
$$E_0^{(2,2)} = \sum_{Q_1,Q_2=1}^{\infty} Q_1 Q_2 \int_{-\infty}^{\infty}\frac{d\tilde p_1}{2\pi}e^{-L\tilde\varepsilon_{Q_1}(\tilde p_1)}\int_{-\infty}^{\infty}\frac{d\tilde p_2}{2\pi}e^{-L\tilde\varepsilon_{Q_2}(\tilde p_2)}\, i\partial_{\tilde p_1}\Big\{Q_1 Q_2(2-[2]_q)^2(2-[2]_{\dot q})^2 \log S_{sl(2)}^{Q_1 Q_2}(\tilde p_1,\tilde p_2)$$
$$+(2-[2]_{\dot q})^2\Big[[3]_q\left(Q_1 Q_2 \log U_0 U_1 U_2 + \mathcal{K}^{Q_1 Q_2}\right) - [2]_q\Big((4Q_1Q_2 - Q_1 - Q_2)\log U_0 + 2Q_2(2Q_1-1)\log U_1$$
$$+2Q_1(2Q_2-1)\log U_2 + (Q_2-Q_1)\log U_3 + 4\mathcal{K}^{Q_1Q_2}\Big) + [1]_q\Big((5Q_1Q_2 - 2Q_1 - 2Q_2)\log U_0 + Q_2(5Q_1-4)\log U_1$$
$$+Q_1(5Q_2-4)\log U_2 + 2(Q_2-Q_1)\log U_3 + 5\mathcal{K}^{Q_1Q_2}\Big)\Big] + (q\leftrightarrow\dot q)\Big\}$$

Outline

TBA equations for planar AdS/CFT [D.B, Fioravanti, Tateo; Gromov, Kazakov, Kozak, Vieira; Arutyunov, Frolov '09(b)]

$$Z(R,L) = Tr\, e^{-RH(L)} \underset{R\to\infty}{\approx} e^{-RE_0(L)}$$

$$\tilde{Z}(R,L) = Tr\, e^{-L\tilde{H}(R)} \underset{R\to\infty}{\approx} e^{-RLf(L)} \quad \text{with } T = 1/L$$



- taking the logarithm and the thermodynamic limit $R, K_{\mathrm{I}}, K^{\alpha}_{\mathrm{II}}, K^{\alpha}_{\mathrm{III}} \to \infty$ of the mirror BAEs $\Rightarrow$ Bethe roots organize in Bethe strings $\Rightarrow$ BAEs of real centers [Arutyunov, Frolov '09(a)]
  $\Rightarrow$ infinite set of **density equations** for $\rho_Q, \rho^{\alpha}_{y|\pm}, \rho^{\alpha}_{v,K}, \rho^{\alpha}_{w,K}$:

$$\rho^r_A(u) + \rho^h_A(u) = \frac{d\tilde{p}(u)}{du}\delta_{A,Q} + \sum_B K_{AB}(u,v) * \rho_B(v)$$

- minimization of the free energy
  $\Rightarrow$ infinite set of **TBA equations** for the pseudoenergies $\varepsilon_Q, \varepsilon^{\alpha}_{y|\pm}, \varepsilon^{\alpha}_{v,K}, \varepsilon^{\alpha}_{w,K}$

$$\ln\frac{\rho^h_A}{\rho^r_A} \equiv \varepsilon_A(u) = -\mu_Q + LE_Q\delta_{A,Q} - \sum_B \ln(1+e^{-\varepsilon_B(v)}) * K_{BA}(u,v)$$

- exact **ground state energy**

$$E_0(L) = Lf(1/L) = -\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\tilde{p}}{2\pi}\ln(1+e^{-\varepsilon_Q(\tilde{p})}) + i\mu\int_{-\infty}^{\infty}\frac{d\tilde{p}}{2\pi}(\rho^1_{y|+}+\rho^1_{y|-}-\rho^2_{y|+}-\rho^2_{y|-})$$

- SUSY case: $E_0(L) = 0$ (BPS ground state) is determined by Witten's index $Tr(-1)^F e^{-\beta H}$ with $\mu = \pi$

AdS/CFT Y-system [Gromov, Kazakov, Vieira; D.B, Fioravanti, Tateo; Gromov, Kazakov, Kozak, Vieira; Arutyunov, Frolov '09]

- acting on the TBA equations with the operator $(K+1)^{-1}_{MN} = \delta_{MN} - s(I_{MN})$, with $I_{MN} = \delta_{N,M+1} + \delta_{N,M-1}$, $s(u) = \frac{g}{4\cosh(g\pi u/2)}$, $s(u+i/g-i0) + s(u-i/g+i0) = \delta(u)$
  $\Rightarrow$ universal or "simplified" TBA equations [Zamolodchikov '91; Arutyunov, Frolov '09]
- or using directly [Ravanini, Tateo, Valleriani '92]

$$K_{KM}\left(u, v+\frac{i}{g}-i0\right) + K_{KM}\left(u, v-\frac{i}{g}+i0\right) - \sum_{K'=1}^{\infty} I_{KK'} K^{K'M}(u,v) = -I_{KM}\delta(u-v)$$

$$\Rightarrow \varepsilon_K\left(u+\frac{i}{g}\right) + \varepsilon_K\left(u-\frac{i}{g}\right) + \sum_{K'=1}^{\infty} I_{KK'}\varepsilon_{K'}(u) = \sum_{K'=1}^{\infty} I_{KK'} \ln(1+e^{-\varepsilon_{K'}(u)})$$

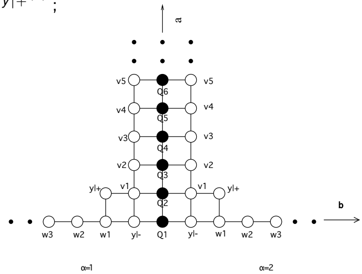
and defining

$$Y_{Q,0}(u) = e^{-\varepsilon_Q(u)}\,;\; Y_{1,\pm 1}(u) = e^{\varepsilon^{1,2}_{y|-}(u)}\,;\; Y_{2,\pm 2}(u) = e^{\varepsilon^{1,2}_{y|+}(u)}\,;$$

$$Y_{K+1,\pm 1}(u) = e^{\varepsilon^{1,2}_{v,K}(u)}\,;\; Y_{1,\pm K-1}(\lambda) = e^{\varepsilon^{1,2}_{w,K}(u)}$$

$$\Rightarrow \qquad Y^+_{a,b}Y^-_{a,b} = \frac{(1+Y_{a+1,b})(1+Y_{a-1,b})}{\left(1+Y^{-1}_{a,b+1}\right)\left(1+Y^{-1}_{a,b-1}\right)}$$

Y-system + Y-functions' analytic properties $\Rightarrow$ ground-state TBA [Cavagliá, Fioravanti, Tateo '10]
Y-system + Y-functions' analytic properties $\Rightarrow sl(2)$ excited states TBA [Balog, Hegedus '11]



TBA equations for deformed AdS/CFT

- **twist**: a **defect** line with transmission matrix $T = e^{2i\gamma_l h} \otimes e^{2i\gamma_r h}$
  $\Rightarrow$ **defect operator** in the *mirror* theory partition function as a **chemical potential**

$$Z(L,R) = \sum_n e^{-L\tilde{E}_n(R)+2i\gamma_\alpha h^\alpha} = \sum_{\alpha=l,r} \sum_A \int d[\rho_A] e^{-L\tilde{E}[\rho_Q]+\mu[\rho^\alpha_{y|\pm},\rho^\alpha_{M,w}]+S[\rho^\alpha_A,\bar{\rho}^\alpha_A]}$$

where $e^{2i\gamma_\alpha h_\alpha}|K^{\mathrm{I}}_\alpha, K^{\mathrm{II}}_\alpha, K^{\mathrm{III}}_\alpha >= e^{i\gamma_\alpha(K^{\mathrm{II}}_\alpha - 2K^{\mathrm{III}}_\alpha)}|K^{\mathrm{I}}_\alpha, K^{\mathrm{II}}_\alpha, K^{\mathrm{III}}_\alpha >$

$$\Rightarrow \mu[\rho^\alpha_{y|\pm},\rho^\alpha_{M,w}] = R \sum_{\alpha=1,2} \int du \left[\mu^\alpha_y(\rho^\alpha_{y|-} + \rho^\alpha_{y|+})(u) + \sum_{M=1}^{\infty} \mu^\alpha_{w|M}\rho^\alpha_{w|M}(u) \right]$$

- undeformed mirror BAEs
- twisted **TBA** equations

$$\epsilon_{Q_2} = L\tilde{E}_{Q_2} - \log(1+e^{-\epsilon_{Q_1}}) \star K_{Q_1 Q_2} + \sum_{\alpha=l,r} \log(1+e^{-\epsilon^\alpha_{v|M}}) \star K_{v|M\, Q_2} \mp \log(1-e^{-\epsilon^\alpha_{y|\pm}}) \star K_{y|\pm\, Q_2}$$

$$\epsilon^\alpha_{v|M} = -\log(1+e^{-\epsilon_{Q_2}}) \star K_{Q_2\, v|M} + \log(1+e^{-\epsilon^\alpha_{v|M'}}) \star K_{v|M'\, v|M} \mp \log(1-e^{-\epsilon^\alpha_{y|\pm}}) \star K_{y|\pm\, v|M} \,,$$

$$\epsilon^\alpha_{w|N} = -\mu^\alpha_{w|N} - \log(1+e^{-\epsilon^\alpha_{w|N'}}) \star K_{w|N'\, w|N} \mp \log(1-e^{-\epsilon^\alpha_{y|\pm}}) \star K_{y|\pm\, w|N} \,,$$

$$\epsilon^\alpha_{y|\pm} = -\mu^\alpha_y - \log(1+e^{-\epsilon_{Q_2}}) \star K_{Q_2\, y|\pm} + \log(1+e^{-\epsilon^\alpha_{v|M}}) \star K_{v|M\, y|\pm} - \log(1+e^{-\epsilon^\alpha_{w|N}}) \star K_{w|N\, y|\pm}$$

where $\mu^{l,r}_y = i\gamma_{l,r}\,;\quad \mu^{l,r}_{w|M} = -2Mi\gamma_{l,r}\,;\quad \gamma_l = (\gamma_3-\gamma_2)L/2\,;\ \gamma_r = (\gamma_3+\gamma_2)L/2$

Hybrid TBA equations for (un)deformed AdS/CFT

- $(2\delta_{MN} - I_{MN})\mu^{\alpha}_{M|w} = 0$; $2\mu^{\alpha}_{y} - \mu^{\alpha}_{1|w} = 0 \Rightarrow$ **simplified** TBA equations and **Y-system** are the same as the **undeformed** ones [Gromov, Levkovich-Maslyuk '10; de Leeuw, van Tongeren '11]
- for convenience, unsimplified or canonical TBA equation for massive Y_Q functions $\Rightarrow$ "hybrid" TBA equations [Arutyunov, Frolov, Suzuki '09]

$$\ln Y_Q = -L\,\tilde{E}_Q + \ln(1+Y_{Q'}) * K^{Q'Q}_{sl(2)} + \sum_{\alpha=1,2} \ln(1+1/Y^{\alpha}_{v|M}) * K^{MQ}_{vx}$$

$$+\frac{1}{2}\sum_{\alpha=1,2}\left[\ln\frac{1-1/Y^{\alpha}_{-}}{1-1/Y^{\alpha}_{+}} \hat{*} K_Q + \ln(1-1/Y^{\alpha}_{-})(1-1/Y^{\alpha}_{+}) \hat{*} K_{yQ}\right] ,$$

$$\ln Y^{\alpha}_{-} = \ln\frac{1+Y^{\alpha}_{v|1}}{1+Y^{\alpha}_{w|1}} * s - \ln(1+Y_Q) * K^{-}_{Qy} + \ln(1+Y_Q) * K^{Q1}_{xv} * s ,$$

$$\ln\frac{Y^{\alpha}_{+}}{Y^{\alpha}_{-}} = \ln(1+Y_Q) * K_{Qy} ,$$

$$\ln Y^{\alpha}_{v|M} = I_{MN} \ln(1+Y^{\alpha}_{v|N}) * s + \delta_{M1} \ln\frac{1-Y^{\alpha}_{-}}{1-Y^{\alpha}_{+}} \hat{*} s - \ln(1+Y_{M+1}) * s ,$$

$$\ln Y^{\alpha}_{w|M} = I_{MN} \ln(1+Y^{\alpha}_{w|N}) * s + \delta_{M1} \ln\frac{1-1/Y^{\alpha}_{-}}{1-1/Y^{\alpha}_{+}} \hat{*} s .$$

where $Y_Q = e^{-\varepsilon_Q}$; $Y_A = e^{\varepsilon_A}$; $I_{MN} = \delta_{N,M+1} + \delta_{N,M-1}$

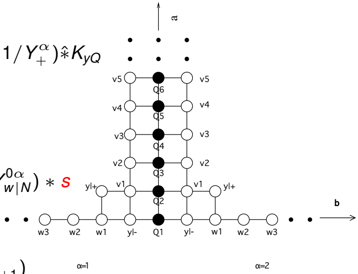
LO TBA equations and corrections

asymptotic expansion $Y_A = Y_A^0(1 + y_A)$

$$\ln Y_Q^0 = -L\,\tilde{E}_Q + \sum_{\alpha=1,2} \ln(1 + 1/Y^{0\alpha}_{v|M}) * K^{MQ}_{vx} + \frac{1}{2}\ln(1 - 1/Y^{\alpha}_{-})(1 - 1/Y^{\alpha}_{+})\,\hat{*}K_{yQ}$$

$$\ln Y^{0\alpha}_{-} = \ln\frac{1 + Y^{0\alpha}_{v|1}}{1 + Y^{0\alpha}_{w|1}} * s\,, \qquad \ln\frac{Y^{0\alpha}_{+}}{Y^{0\alpha}_{-}} = 0\,,$$

$$\ln Y^{0\alpha}_{v|M} = I_{MN}\ln(1 + Y^{0\alpha}_{v|N}) * s\,, \qquad \ln Y^{0\alpha}_{w|M} = I_{MN}\ln(1 + Y^{0\alpha}_{w|N}) * s$$



both magnonic Y-functions satisfy the same equation

$$(Y^0_M)^2 = (1 + Y^0_{M-1})(1 + Y^0_{M+1})$$

but different **asymptotic conditions**

$$\frac{\ln Y^{0\alpha}_{v|M}}{M} \to_{M\to\infty} 0\,; \qquad \frac{\ln Y^{0\alpha}_{w|M}}{M} \to_{M\to\infty} -\mu^{\alpha}_{w|M} = 2iM\gamma_\alpha$$

$$\Rightarrow\; Y^{0\alpha}_{v|M} = M(M+2)\,; \qquad Y^{0L}_{w|M} = [M]_q[M+2]_q\,; \qquad Y^{0R}_{w|M} = [M]_{\dot{q}}[M+2]_{\dot{q}}$$

leading order finite-size correction for the vacuum

$$Y^0_Q \simeq Q^2\,(2 - [2]_q)\,(2 - [2]_{\dot{q}})\,e^{-L\,\tilde{E}_Q}$$

$$\delta E = -\sum_{Q=1}^{\infty}\int\frac{d\tilde{p}^Q}{2\pi}\ln(1 + Y_Q) \simeq -\sum_{Q=1}^{\infty}\int\frac{d\tilde{p}^Q}{2\pi}Y^0_Q + \sum_{Q=1}^{\infty}\int\frac{d\tilde{p}^Q}{2\pi}\frac{(Y^0_Q)^2}{2}$$

NLO TBA equations

solution of the linearized set of hybrid TBA equations:

- magnonic equations for y_w and y_v : difference eqs. with driving term $y_+^\alpha - y_-^\alpha = Y_Q^0 \star K_{Qy}$
- solution of the "horizontal" part of the Y-system

$$y_{w|N}^\alpha = \left(\frac{[N-1]_\alpha [N+1]_\alpha}{[N]_\alpha^2} y_{w|N-1}^\alpha + \frac{[N+1]_\alpha [N+3]_\alpha}{[N+2]_\alpha^2} y_{w|N+1}^\alpha \right) \star s + \delta_{N1} \frac{y_+^\alpha - y_-^\alpha}{1 - Y_+^0} \hat{\star} s$$

$$\Rightarrow y_{w|1}^\alpha = \frac{[2]_\alpha}{[2]_\alpha - 2} Y_1^0 \star K_{1y} \hat{\star} \left(K_1 - \frac{1}{[3]_\alpha} K_3 \right)$$

- solution of the "vertical" part of the Y-system: inhomogeneous term $Y_{M+1}^0 \star s$ for any M

$$y_{v|M}^\alpha = \left(\frac{(M-1)(M+1)}{M^2} y_{v|M-1}^\alpha + \frac{(M+1)(M+3)}{(M+2)^2} y_{v|M+1}^\alpha \right) \star s - Y_{M+1}^0 \star s + \delta_{M1} \frac{y_-^\alpha - y_+^\alpha}{1 - \frac{1}{Y_+^{0\alpha}}} \hat{\star} s$$

$$\Rightarrow y_{v|M}^\alpha = \frac{(M+1) Y_Q^0}{QM(M+2)} \star \sum_{k=0}^{Q-1} k(k-Q) \left[(M+2) K_{M-Q+2k} - M K_{M-Q+2k+2} \right] + \frac{Y_Q^0 \star K_{Qy} \hat{\star} \left[M K_{M+2} - (M+2) K_M \right]}{(M+1)(2-[2])}$$

$$y_+^\alpha + y_-^\alpha = 2 \left(\frac{3}{4} y_{v|1}^\alpha - \frac{[3]_\alpha}{[2]_\alpha^2} y_{w|1}^\alpha \right) \star s - Y_Q^0 \star K_Q + 2 Y_Q^0 \star K_{xv}^{Q1} \star s$$

$$y_{Q_2} = Y_{Q_1}^0 \star \left(K_{sl(2)}^{Q_1 Q_2} + 2s \star K_{vx}^{Q_1 - 1, Q_2} \right) + \sum_{\alpha = l,r} \left[\frac{3}{4} y_{v|1}^\alpha \star s \hat{\star} K_{yQ_2} + \frac{Q_2^2 - 1}{Q_2^2} y_{v|Q_2 - 1}^\alpha \star s \right.$$
$$\left. - \frac{y_-^\alpha - y_+^\alpha}{1 - 1/Y_+^{0\alpha}} \hat{\star} s \star K_{vx}^{1 Q_2} + \frac{y_-^\alpha - y_+^\alpha}{2(Y_+^{0\alpha} - 1)} \hat{\star} K_{Q_2} + \frac{y_-^\alpha + y_+^\alpha}{2(Y_+^{0\alpha} - 1)} \hat{\star} K_{yQ_2} \right]$$

NLO TBA corrections

inserting all the terms and simplifying: final result

$$
\begin{aligned}
y_{Q_2} &= Y^0_{Q_1} \star \frac{1}{2\pi i}\partial_{u_1}\Big\{ \log S^{Q_1 Q_2}_{sl(2)} + \frac{2}{Q_1 Q_2}\mathcal{K}^{Q_1 Q_2} - 2\log a_1^{Q_1 Q_2}(u_1, u_2) \\
&- \sum_{\alpha=l,r} \frac{1}{(2-[2]_\alpha)}\left[\frac{1}{Q_2}\log a_2^{Q_1 Q_2}(u_1, u_2) + \frac{1}{Q_1}\log a_2^{Q_2 Q_1}(u_2, u_1)^\star\right]\Big\}
\end{aligned}
$$

where

$$a_1^{Q_1 Q_2} = (U_0 U_1 U_2)^{-1}\,, \quad a_2^{Q_1 Q_2} = U_0 U_2^2 U_3\,, \quad a_2^{Q_2 Q_1 *} = U_0 U_1^2 U_3^{-1}$$

energy expression $E_0^{(2,2)}(L) = -\sum_{Q=1}^{\infty}\int \frac{d\tilde{p}}{2\pi} Y_Q^0 y_Q$

scalar factor matching Lüscher:

$$(2-[2]_q)^2(2-[2]_{\dot{q}})^2 \sum_{Q_1,Q_2=1}^{\infty} Q_1^2 Q_2^2 \int_{-\infty}^{\infty}\frac{d\tilde{p}_1}{2\pi} e^{-L\tilde{\epsilon}_{Q_1}(\tilde{p}_1)}\int_{-\infty}^{\infty}\frac{d\tilde{p}_2}{2\pi} e^{-L\tilde{\epsilon}_{Q_2}(\tilde{p}_2)} i\partial_{\tilde{p}_1}\log S^{Q_1 Q_2}_{sl(2)}(\tilde{p}_1, \tilde{p}_2)$$

intermediate check: cases (Q,1) and (1,Q) are related by complex conjugation and $1 \leftrightarrow 2$

$\Leftrightarrow$ unitarity $S^{1Q}(u_1, u_2)S^{Q1}(u_{2,u_1}) = \mathrm{I} \Leftrightarrow a_k^{Q1}(u_1, u_2) = a_k^{1Q}(u_2^*, u_1^*)^*$

the complete result exactly matches the NLO Lüscher correction!

Weak coupling expansion

- $x^{\pm}$ mirror: $x^{\pm}(\tilde{p}) = \frac{(\tilde{p}-iQ)}{2g}\left(\sqrt{1+\frac{4g^2}{Q^2+\tilde{p}^2}} \mp 1\right)$; $x^{-} = \frac{\tilde{p}-iQ}{g} + O(g)$; $x^{+} = \frac{g}{\tilde{p}+iQ} + O(g^3)$
- **single wrapping**: $e^{-L\tilde{\epsilon}_Q(\tilde{p})} = \sum_{j=0}^{\infty} c_j \frac{g^{2(L+j)}}{(\tilde{p}^2+Q^2)^{L+j}}$

$$-(2-[2]_q)(2-[2]_{\dot{q}})\sum_{Q=1}^{\infty} Q^2 \int \frac{d\tilde{p}}{2\pi} e^{-L\tilde{\epsilon}_Q(\tilde{p})} = -(2-[2]_q)(2-[2]_{\dot{q}})\sum_{j=0}^{\infty} c_j 2^{1-2(L+j)} \begin{pmatrix} 2(L+j-1) \\ L+j-1 \end{pmatrix} \zeta_{2(L+j)-3}\, g^{2(L+j)}$$

- **double wrapping**:
  - quadratic term

$$\frac{1}{2}(2-[2]_q)^2(2-[2]_{\dot{q}})^2 \sum_{Q=1}^{\infty} Q^4 \int \frac{d\tilde{p}}{2\pi} e^{-2L\tilde{\epsilon}_Q(\tilde{p})} = (2-[2]_q)^2(2-[2]_{\dot{q}})^2 2^{-4L} \begin{pmatrix} 4L-2 \\ 2L-1 \end{pmatrix} \zeta_{4L-5}\, g^{4L}$$

  - pure NLO term $E_0^{(2,2)}(L) = -\sum_{Q=1}^{\infty} \int \frac{d\tilde{p}}{2\pi} Y_Q^0 y_Q$
  - the **dressing phase** contributes at LO weak coupling:

$$\frac{1}{2\pi i}\partial_{\tilde{p}_1} \log S_{sl(2)}^{Q_1 Q_2}(\tilde{p}_1,\tilde{p}_2) \simeq \frac{1}{4\pi}\Big[\psi\Big(\frac{1}{2}(i(\tilde{p}_2-\tilde{p}_1)-Q_1+Q_2)\Big)+\psi\Big(1+\frac{1}{2}(i(\tilde{p}_2-\tilde{p}_1)-Q_1+Q_2)\Big)$$
$$+\psi\Big(\frac{1}{2}(i(\tilde{p}_2-\tilde{p}_1)+Q_1+Q_2)\Big)+\psi\Big(1+\frac{1}{2}(i(\tilde{p}_2-\tilde{p}_1)+Q_1+Q_2)\Big)-2\psi\Big(1-\frac{i}{2}(\tilde{p}_1+iQ1)\Big)+c.c\Big]$$

- $L=3$ (TrZ^3) result (total an. dim.):

$$E_0(3) = -(2-[2]_q)(2-[2]_{\dot{q}})\left(\frac{3}{16}\zeta_3 g^6 - \frac{15}{16}\zeta_5 g^8 + \frac{945}{256}\zeta_7 g^{10} - \frac{3465}{256}\zeta_9 g^{12}\right)$$
$$-(2-[2]_q)(2-[2]_{\dot{q}})\left([2]_q+[2]_{\dot{q}}-4\right)\frac{15}{256}\zeta_3\zeta_5 g^{12} + (2-[2]_q)^2(2-[2]_{\dot{q}})^2\left(-\frac{9}{256}\zeta_3^2+\frac{189}{4096}\zeta_7\right)g^{12}+\dots$$

Outline

Results...

- we have studied and applied techniques and ideas of the **integrable models** to marginal non-supersymmetric deformations of the **AdS_5/CFT_4 correspondence**
- how to **twist** the asymptotic **S-matrix**: Reshetikhin twist + twisted boundary conditions
- we derived **LO** and **NLO vacuum Lüscher-like corrections**, involving the bound-states mirror S-matrix
- for the vacuum only twisted boundary conditions are relevant
- we have formulated the **ground-state TBA** equations of the **deformed** AdS_5/CFT_4
- solution of **ground-state twisted TBA** equations at NLO
- vacuum finite-size corrections for the non-supersymmetric case: matching TBA-Lüscher at leading and next-to-leading order (large volume)
  $\Rightarrow$ check of the bound-states mirror S-matrix and the AdS_5/CFT_4 TBA equations
- explicit calculation of single and double-wrapping corrections
  $\Rightarrow$ prediction for future **perturbative** diagrammatic gauge calculations

...and Outlook

- **strong coupling** limit?
- group theoretical (SU(2|2) Yangian symmetry) formulation for the determinants of the bound-states S-matrices in all the $su(2)_L \otimes su(2)_R$ sectors?
- divergence for $L = 2$ (as for the undeformed case)!
- higher wrappings and subleading contributions at weak coupling
- to study **excited states TBA** equations for the **SU(2)** and **SL(2)** sector
  $\Rightarrow$ check of the known results (Lüscher and diagrammatic) at weak coupling
  $\Rightarrow$ check of the conjectured relation with the **orbifolded** theory
- **TBA for all the excited states** $\Rightarrow$ exact deformed AdS/CFT spectra
- exact solution at **any** g (numerically or analytically) of β-deformed ($\mathcal{N} = 1$) and non-supersymmetric SYM $\Rightarrow$ closer to **QCD**
- understanding twisted theory important for recent developments towards construction of AdS/CFT Q-operators [Bazhanov, Frassek, Lukowski, Meneghelli, Staudacher. '10]: twist factors as **regulators**

THANK YOU !