Finite size Emptiness Formation Probability for the XXZ spin chain at $\Delta = -1/2$

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XXZ spin chain at $\Delta = -\frac{1}{2}$

Hamiltonian of the XXZ spin chain

$$H_N(\Delta) = -\frac{1}{2}\sum_{i=1}^{N} \sigma_x^i \sigma_x^{i+1} + \sigma_y^i \sigma_y^{i+1} + \Delta \sigma_z^i \sigma_z^{i+1} \qquad \Delta = -\frac{q+q^{-1}}{2}$$

[Razumov, Stroganov '00] $\Delta = -\frac{1}{2} \to$ g.s. energy $E = -\frac{3N}{4}$

- N odd ($N = 2n+1$)
  Periodic boundary conditions: $\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}$.
  Two-fold degenerate ground state $\Psi_{2n+1}^{\pm}$, rational-valued components.
- N even ($N = 2n$)
  Twisted periodic boundary conditions: $\sigma_{N+1}^{z} = \sigma_1^{z}$, while $\sigma_{N+1}^{\pm} = e^{\pm i\frac{2\pi}{3}}\sigma_{N+1}^{\pm}$, where $\sigma^{\pm} = \sigma^x + \pm i\sigma^y$.
  A single ground state Ψ_{2n}^{e}, complex valued components.

Emptiness Formation Probability (EFP)

The definition of the EFP

$$E_N^\mu(k) = \frac{\left((\Psi_N^\mu)^*, \prod_{i=1}^k p_i^+ \cdot \Psi_N^\mu\right)}{(\Psi_N^\mu, \Psi_N^\mu)}, \qquad p_i^+ = \frac{\sigma_i^z + \mathbf{1}}{2}$$

$$\mu = \pm, e$$

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we define also a "Pseudo" EFP

$$E_{2n}^{\tilde{e}}(k) = \frac{\left(\Psi_{2n}^{e}, \prod_{i=1}^{k} p_i^{+} \cdot \Psi_{2n}^{e}\right)}{(\Psi_{2n}^{e}, \Psi_{2n}^{e})}.$$

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They have an analytical, "factorized" form!

Conjectures [Razumov, Stroganov '00]

$$\frac{E^-_{2n+1}(k-1)}{E^-_{2n+1}(k)} = \frac{(2k-2)!(2k-1)!(2n+k)!(n-k)!}{(k-1)!(3k-2)!(2n-k+1)!(n+k-1)!}$$

$$\frac{E^+_{2n+1}(k-1)}{E^+_{2n+1}(k)} = \frac{(2k-2)!(2k-1)!(2n+k)!(n-k+1)!}{(k-1)!(3k-2)!(2n-k+1)!(n+k)!}$$

$$\frac{E^e_{2n}(k-1)}{E^e_{2n}(k)} = \frac{(2k-2)!(2k-1)!(2n+k-1)!(n-k)!}{(k-1)!(3k-2)!(2n-k)!(n+k-1)!}$$

$$\frac{E^{\tilde{e}}_{2n}(k-1)}{E^{\tilde{e}}_{2n}(k)} = -q\frac{(2k-2)!(2k-1)!(2n+k-1)!(n-k)!}{(k-1)!(3k-3)!(3k-1)(2n-k)!(n+k-1)!}$$

Some partial results

- Asymptotics $N \to \infty$ [Maillet et al. '02]

$$\lim_{N\to\infty} E_N(k) = \left(\frac{\sqrt{3}}{2}\right)^{3k^2} \prod_{j=1}^{k} \frac{\Gamma(j-1/3)\Gamma(j+1/3)}{\Gamma(j-1/2)\Gamma(j+1/2)}.$$

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- Norm [Di Francesco et al. '06]

$$E^{\pm}_{2n+1}(n) = A_{HT}(2n+1)^{-1} = \prod_{j=1}^{n} \frac{(2j-1)!^2(2j)!^2}{(j-1)!j!(3j-1)!(3j)!}$$

$$E^{\tilde{e}}_{2n}(n) = (-q)^n A_n^{-2} = (-q)^n CSSC(2n)^{-1}$$

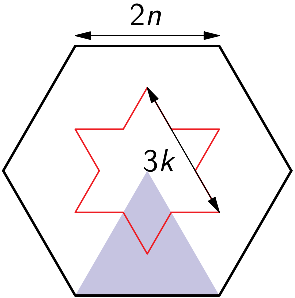
where:

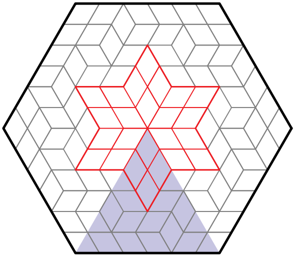
- $A_n = \#$ of $n \times n$ Alternating Sign Matrices
- $A_{HT}(2n+1) = \#$ of $(2n+1) \times (2n+1)$ Half-Turn Symmetric Alternating Sign Matrices

$CSSC(2n, k-1)$

$$\frac{E^{\tilde{e}}_{2n}(k-1)}{E^{\tilde{e}}_{2n}(k)} = -q\frac{CSSC(2n,k-1)}{CSSC(2n,k)}$$

$CSSC(2n, k-1) = \#$ of lozange tilings of a regular hexagon of side length $2n$, which are invariant under a rotation of $\pi/3$ and have a star shaped frozen region of length k,
$(CSSC(2n) := CSSC(2n, 0) = CSSC(2n, 1))$





Integrability

R-matrix

$$R_{i,j}(z) = \begin{pmatrix} a(z) & 0 & 0 & 0 \\ 0 & b(z) & c_1(z) & 0 \\ 0 & c_2(z) & b(z) & 0 \\ 0 & 0 & 0 & a(z) \end{pmatrix}$$

with

$$a(z) = \frac{qz - q^{-1}}{q - q^{-1}z}, \qquad b(x) = \frac{z-1}{q - q^{-1}z},$$

$$c_1(z) = \frac{(q - q^{-1})z}{q - q^{-1}z}, \qquad c_2(z) = \frac{(q - q^{-1})}{q - q^{-1}z}.$$

Twist matrix

$$\Omega(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

R-Ω commutation

$$[R_{i,j}(x), \Omega_i(\phi) \otimes \Omega_j(\phi)] = 0$$

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Yang-Baxter equation

$$R_{i,j}(z_i/z_j)R_{i,k}(z_i/z_k)R_{j,k}(z_j/z_k) = R_{j,k}(z_j/z_k)R_{i,k}(z_i/z_k)R_{i,j}(z_i/z_j)$$

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Transfer matrix

$$T_N(y|\mathbf{z}_{\{1,\ldots,N\}}, \phi) = \mathrm{tr}_0 \left[R_{0,1}(y/z_1)R_{0,2}(y/z_2)\ldots R_{0,N}(y/z_N)\Omega_0(\phi)\right]$$

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Crucial idea!

Consider the ground state with spectral parameters

$$\Psi_N^\mu \rightarrow \Psi_N^\mu(\mathbf{z}), \qquad T_N(y|\mathbf{z}_{\{1,\ldots,N\}}, \phi)\Psi_N^\mu(\mathbf{z}) = 1\Psi_N^\mu(\mathbf{z})$$

quantum Knizhnik–Zamolodchikov (qKZ) equations

[Razumov, Stroganov & Zinn-Justin '07]

The ground state satisfies a specialization $q = e^{2\pi i/3}$ of the $U_q(sl_2)$ qKZ equations

$$\check{R}_{i,i+1}(z_{i+1}/z_i)\Psi_N^\mu(\dots, z_i, z_{i+1}, \dots) = \Psi_N^\mu(\dots, z_{i+1}, z_i, \dots)$$

$$\sigma\Psi_N^\mu(z_1, z_2, \dots, z_{N-1}, z_N) = D\Psi_N^\mu(z_2, \dots, z_{N-1}, z_N, sz_1).$$

With

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_{N-1} \otimes v_N) = v_2 \otimes \cdots \otimes v_{N-1} \otimes v_N \otimes v_1$$

$$\check{R}_{i,i+1}(z) = P_{i,i+1}R_{i,i+1}(z) \qquad P_{i,j}(e_i^\mu \otimes e_j^\nu) = e_i^\nu \otimes e_j^\mu,$$

$$s = q^6, \qquad D = q^{3N} q^{3(s_N^z+1)/2}.$$

Inhomogenous EFP

Using the solution of the $U_q(sl_2)$ qKZ equations at level 1.

we define an inhomogenous version of the EFP

$$\mathcal{E}_N^{\mu}(k; \mathbf{y}_{\{1,\dots,2k\}}; \mathbf{z}_{\{1,\dots,N-k\}}) \sim$$

$$\frac{\left(\mathcal{P}_{N,k}(\mathbf{z})(\Psi_N^{\mu}(q^{-6}\mathbf{y}_{\{k+1,\dots,2k\}};\mathbf{z}))^*, \prod_{i=1}^k p_i^+ \Psi_N^{\mu}(\mathbf{y}_{\{1,\dots,k\}};\mathbf{z})\right)}{\prod_{1\le i<j\le k}(qy_i - q^{-1}y_j)(qy_{i+k} - q^{-1}y_{j+k})}$$

with $\mathcal{P}_{N,k}(\mathbf{z}) = \prod_{i=k+1}^N (z_i p_i^+ + p_i^-)$.

and of the Pseudo EFP

$$\mathcal{E}_N^{\tilde{e}}(k; \mathbf{y}_{\{1,\dots,2k\}}; \mathbf{z}_{\{1,\dots,N-k\}}) \sim$$

$$\frac{\left(\Psi_N^{\mu}(k;q^{-6}\mathbf{y}^{-1}_{\{k+1,\dots,2k\}};\mathbf{z}^{-1}), \prod_{i=1}^k p_i^+ \Psi_N^{\mu}(k;\mathbf{y}_{\{1,\dots,k\}};\mathbf{z})\right)}{\prod_{1\le i<j\le k}(qy_i - q^{-1}y_j)(qy_{i+k} - q^{-1}y_{j+k})}$$

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We shall compute these and then take the specialization $z_i = y_j = 1$.

Properties of $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and of $\mathcal{E}_{2n}^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$

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2. Specialization $z_i = 0$ or $z_i \to \infty$

$$\lim_{z_{2n+1}\to\infty} z_{2n+1}^{-2n}\mathcal{E}_{2n+1}^{-}(k;\mathbf{y};\mathbf{z}) \sim \mathcal{E}_{2n}^{e}(k;\mathbf{y};\mathbf{z}\setminus z_{2n+1})$$

$$\mathcal{E}_{2n+2}^{e}(k;\mathbf{y};\mathbf{z})|_{z_{2n+2}=0} \sim \mathcal{E}_{2n+1}^{+}(k;\mathbf{y};\mathbf{z}\setminus z_{2n+2})$$

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3. Recursion relation for $z_j = q^{\pm 2} z_i$

$$\frac{\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})|_{z_j=q^2 z_i}}{\mathcal{E}_{N-2}^{\mu}(k;\mathbf{y};\mathbf{z}\setminus\{z_i,z_j\})} \sim$$

$$z_i \prod_{\ell=1}^{2k} \frac{q y_\ell - q^{-1} z_i}{q-q^{-1}} \prod_{\substack{1\le \ell \le N-k \\ \ell \ne i,j}} \frac{(q z_\ell - q^{-1} z_i)(q^3 z_i - q^{-1} z_\ell)}{-(q-q^{-1})^2}$$

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$$\prod_{\ell=1}^{2k}\frac{qy_\ell - q^{-1}z_i}{q-q^{-1}}\prod_{\substack{1\le\ell\le 2n-k\\ \ell\neq i,j}}\frac{(qz_\ell - q^{-1}z_i)(q^3 z_i - q^{-1}z_\ell)}{-(q-q^{-1})^2}$$

Properties of $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and of $\mathcal{E}_N^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$

4 Factorized cases

$$\mathcal{E}_{2k}^{e/\tilde{e}}(k;\mathbf{y};\mathbf{z}) = \prod_{1\le i<j\le k} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q-q^{-1})^2}$$

$$\mathcal{E}_{2k+1}^{+}(k+1;\mathbf{y};\mathbf{z}) = \prod_{1\le i<j\le k} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q-q^{-1})^2} \prod_{i=1}^{k} z_i^2$$

$$\mathcal{E}_{2k+1}^{-}(k;\mathbf{y};\mathbf{z}) = \prod_{1\le i<j\le k+1} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q-q^{-1})^2}.$$

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$$\mathcal{E}_{2k+1}^{-}(k;\mathbf{y};\mathbf{z}) = \prod_{1\le i<j\le k+1} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q-q^{-1})^2}.$$

These properties completely determine $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and $\mathcal{E}_{2n}^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$!

Combinatorial point $q = e^{2\pi i/3}$

Let $\tilde{\rho}, \tilde{\sigma}$ be strictly increasing infinite sequences of nonnegative integers. Introduce the following matrices

$$\mathcal{M}^{(\tilde{\rho},\tilde{\sigma})}(r,s;\mathbf{y};\mathbf{z}) = \begin{pmatrix}
z_1^{\tilde{\rho}_1} & z_1^{\tilde{\rho}_2} & \dots & z_1^{\tilde{\rho}_{r+s}} & 0 & 0 & \dots & 0 \\
z_2^{\tilde{\rho}_1} & z_2^{\tilde{\rho}_2} & \dots & z_2^{\tilde{\rho}_{r+s}} & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_r^{\tilde{\rho}_1} & z_r^{\tilde{\rho}_2} & \dots & z_r^{\tilde{\rho}_{r+s}} & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & z_1^{\tilde{\sigma}_1} & z_1^{\tilde{\sigma}_2} & \dots & z_1^{\tilde{\sigma}_{r+s}} \\
0 & 0 & \dots & 0 & z_2^{\tilde{\sigma}_1} & z_2^{\tilde{\sigma}_2} & \dots & z_2^{\tilde{\sigma}_{r+s}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & z_r^{\tilde{\sigma}_1} & z_r^{\tilde{\sigma}_2} & \dots & z_r^{\sigma_{r+s}} \\
y_1^{\tilde{\rho}_1} & y_1^{\tilde{\rho}_2} & \dots & y_1^{\tilde{\rho}_{r+s}} & y_1^{\tilde{\sigma}_1} & y_1^{\tilde{\sigma}_2} & \dots & y_1^{\tilde{\sigma}_{r+s}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
y_{2s}^{\tilde{\rho}_1} & y_{2s}^{\tilde{\rho}_2} & \dots & y_{2s}^{\tilde{\rho}_{r+s}} & y_{2s}^{\tilde{\sigma}_1} & y_{2s}^{\tilde{\sigma}_2} & \dots & y_{2s}^{\tilde{\sigma}_{r+s}}
\end{pmatrix}$$

Combinatorial point $q = e^{2\pi i/3}$

Define the following polynomials

$$\mathcal{S}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z}) = \frac{\det \mathcal{M}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z})}{\prod_{1\le i<j\le r}(z_i-z_j)^2 \prod_{1\le i<j\le 2s}(y_i-y_j) \prod_{\substack{1\le i\le r\\ 1\le j\le 2s}}(z_i-y_j)}.$$

Using the Laplace expansion along the first $r+s$ columns we can write $\mathcal{S}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z})$ as a bilinear in Schur polynomials

$$\mathcal{S}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z}) =$$

$$\sum_{\substack{I\subset\{1,\dots 2s\}\\ |I|=s}} (-1)^{\epsilon(I)} \frac{\prod_{i<j\in I}(y_i-y_j)\prod_{i<j\in I^c}(y_i-y_j)}{\prod_{1\le i<j\le 2s}(y_i-y_j)} S_{\rho(r+s)}(\mathbf{z},\mathbf{y}_I) S_{\sigma(r+s)}(\mathbf{z},\mathbf{y}_{I^c}),$$

Combinatorial point $q = e^{2\pi i/3}$

Now let us introduce the following family of integer sequences

$$\tilde{\lambda}_i(r) = \lfloor \frac{3i-3+r}{2} \rfloor, \qquad \begin{array}{l} \tilde{\lambda}(0) = \{0,1,3,4,6,7,\dots\} \\ \tilde{\lambda}(1) = \{0,2,3,5,6,8,\dots\} \\ \tilde{\lambda}(2) = \{1,2,4,5,7,8,\dots\} \\ \qquad\qquad \cdots \end{array}$$

We claim that

$$\mathcal{E}^{-}_{2n+1}(k;\mathbf{y};\mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(0),\tilde{\lambda}(1))}(2n+1-k,k;\mathbf{y};\mathbf{z})$$

$$\mathcal{E}^{+}_{2n+1}(k;\mathbf{y};\mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(1),\tilde{\lambda}(2))}(2n+1-k,k;\mathbf{y};\mathbf{z})$$

$$\mathcal{E}^{e}_{2n}(k;\mathbf{y};\mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(0),\tilde{\lambda}(1))}(2n-k,k;\mathbf{y};\mathbf{z})$$

$$\mathcal{E}^{\tilde{e}}_{2n}(k;\mathbf{y};\mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(0),\tilde{\lambda}(2))}(2n-k,k;\mathbf{y};\mathbf{z})$$

$t-$ specialization

Setting

$$\mathbf{z} = \{z_1 = t^0, z_2 = t^1, \ldots, z_{N-k} = t^{N-k-1}\}$$

and

$$\mathbf{y} = \{y_1 = t^{N-k}, \ldots, y_{2k} = t^{N+k-1}\}$$

the functions $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and $\mathcal{E}_{2n}^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$, as rational functions of t, have simple factors.

Using the usual t-numbers and t-factorial

$$[i]_t = \frac{t^i - 1}{t - 1} \qquad \text{and} \qquad [n]_t! = \prod_{i=1}^{n} [i]_t.$$

$t-$ specialization: result

$$\frac{\mathcal{E}^{e}_{2n}(k-1;t)}{\mathcal{E}^{e}_{2n}(k;t)} \sim \frac{[2n+k-1]_t![n-k]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k]_t![n+k-1]_{t^3}![k-1]_{t^3}![3k-2]_t!}$$

$$\frac{\mathcal{E}^{\tilde{e}}_{2n}(k-1;t)}{(-q)\mathcal{E}^{\tilde{e}}_{2n}(k;t)} \sim \frac{[2n+k-1]_t![n-k]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k]_t![n+k-1]_{t^3}![k-1]_{t^3}![3k-3]_t![3k-1]_t}$$

$$\frac{\mathcal{E}^{-}_{2n+1}(k-1;t)}{\mathcal{E}^{-}_{2n+1}(k;t)} \sim \frac{[2n+k]_t![n-k]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k+1]_t![n+k-1]_{t^3}![k-1]_{t^3}![3k-2]_t!}$$

$$\frac{\mathcal{E}^{+}_{2n+1}(k-1;t)}{\mathcal{E}^{+}_{2n+1}(k;t)} \sim \frac{[2n+k]_t![n-k+1]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k+1]_t![n+k]_{t^3}![k-1]_{t^3}![3k-2]_t!}$$

The coefficients in front are of the form $t^{\beta}\left(\frac{[3]_t}{3}\right)^{k-1} \xrightarrow[t\to 1]{} 1.$

What if q is generic?

We have found a multiresidua formula

$$\mathcal{E}^{\mu}_{2n-\epsilon(\mu)}(k;\mathbf{y};\mathbf{z}) \sim$$

$$\oint_{\{qz_i\}} \prod_{\ell=1}^{n-k} \frac{dw_\ell w^{\alpha(\mu)} \prod_{j=1}^{2k}(w_\ell - y_j)}{2\pi i \prod_{i=1}^{2n-k-\epsilon(\mu)}(w_\ell - qz_i)(w_\ell - q^{-1}z_i)} K_{n-k}(\mathbf{w})$$

where $\epsilon(\pm) = \pm 1, \epsilon(e) = \epsilon(\tilde{e}) = 0$,

$$\alpha(+) = \alpha(e) = 0, \quad \alpha(\tilde{e}) = 1, \quad \alpha(-) = 2$$

and

$$K_N(\mathbf{w}) = \prod_{1\leq i<j\leq N} (w_i - w_j)^2 (qw_i - q^{-1}w_j)(qw_j - q^{-1}w_i).$$

Perspectives

- Consider other correlation functions.
- Other boundary conditions (e.g. open chain with diagonal $K-$ matrices)
- Higher spins or also $U_q(sl_N)$ spin chain.

Thank you!