

In memory of Alexei Zamolodchikov
Bologna, September 2011

Arctic curves for the six-vertex model

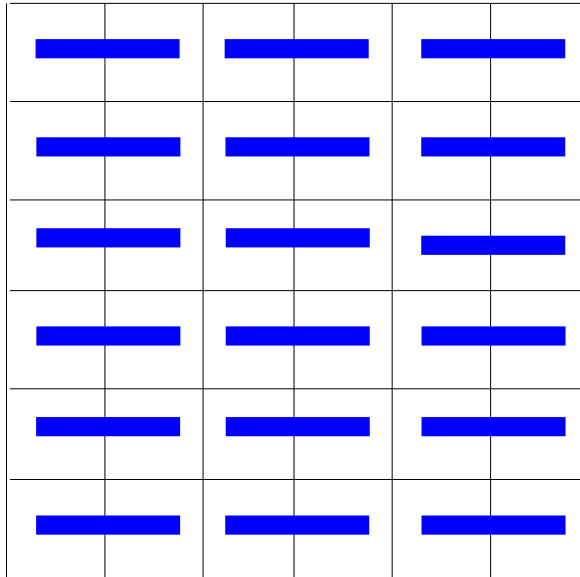
Filippo Colomo
INFN, Firenze

In collaboration with:

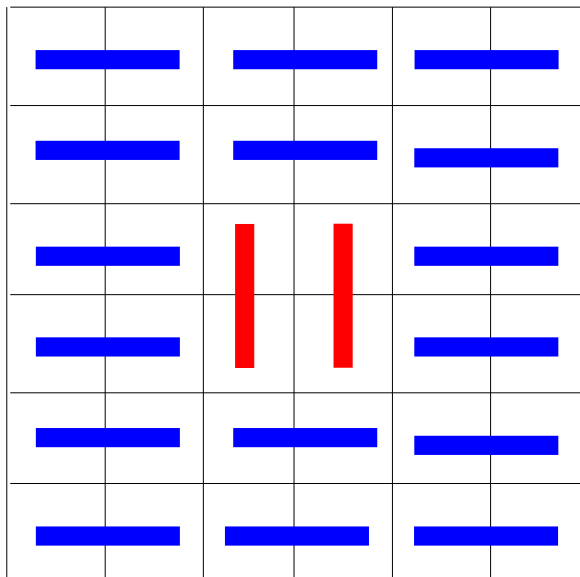
Andrei G. Pronko
Paul Zinn-Justin
Vanni Noferini
Andrea Sportiello

(PDMI Steklov, Saint Petersburg)
(UPMC, Paris)
(Univ. Pisa)
(Univ. Milano)

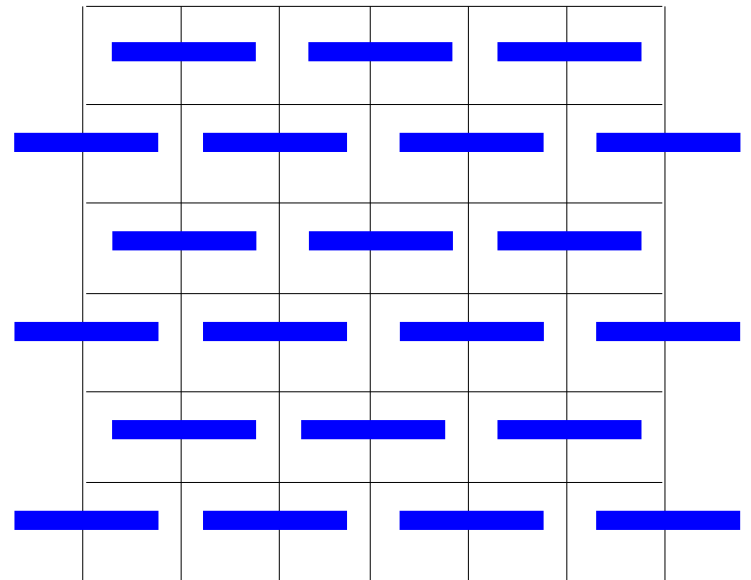
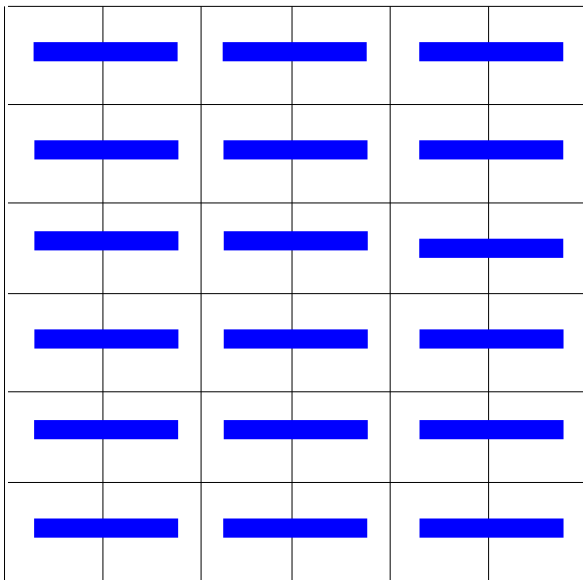
A well-known fact:



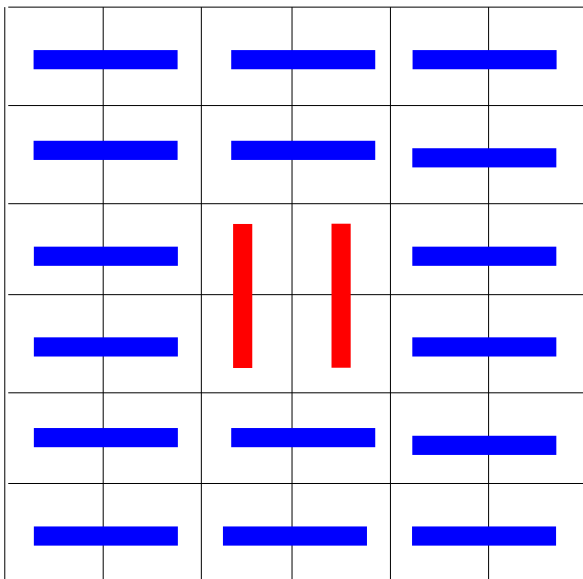
Local defect:



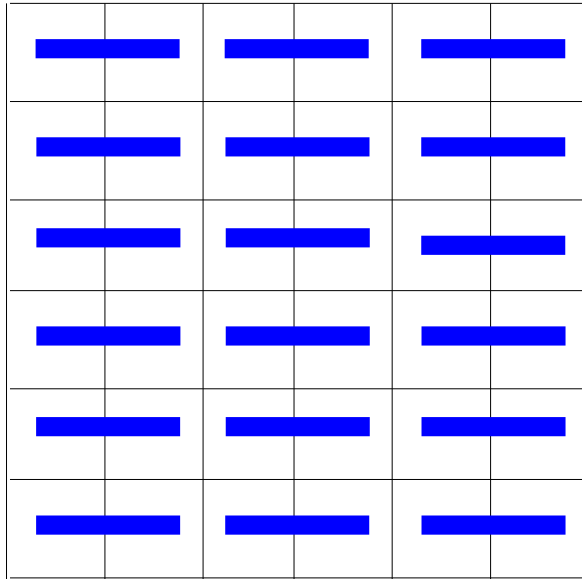
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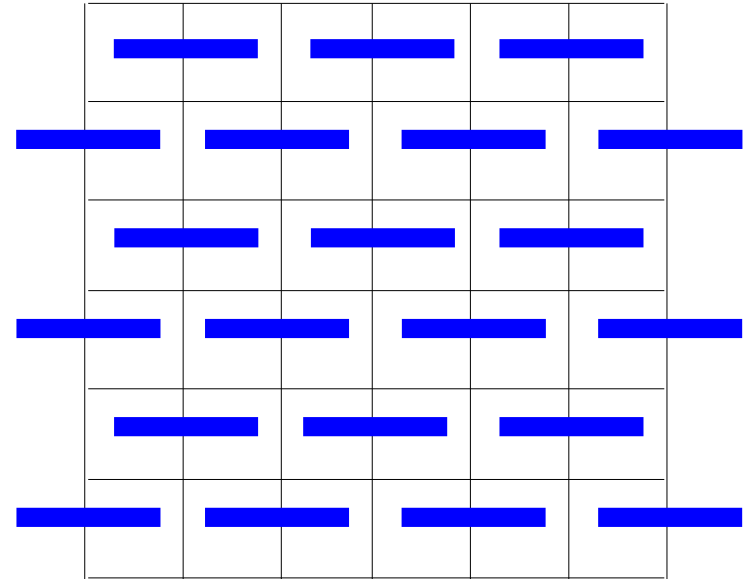
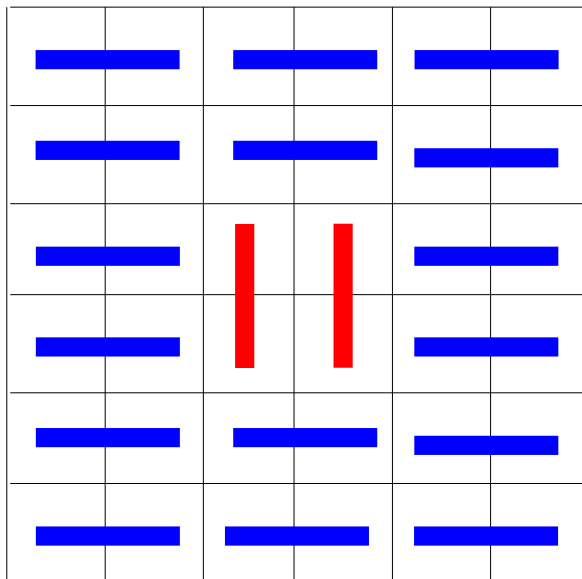
Local defect:



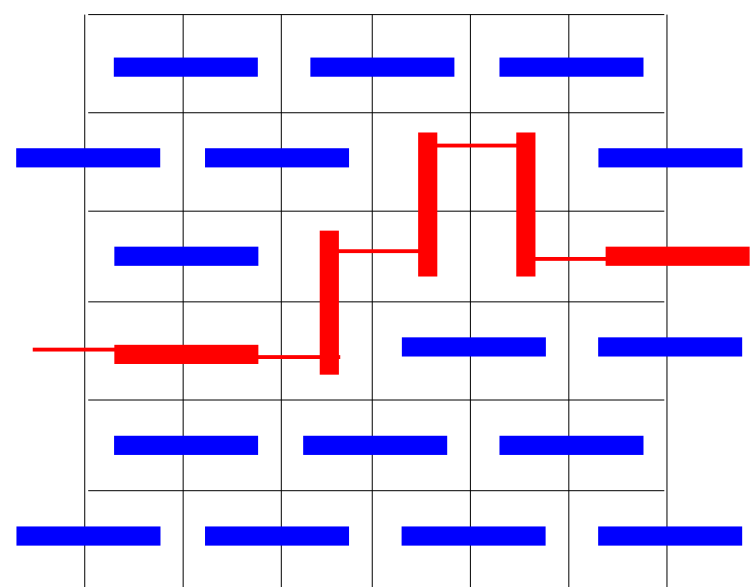
A well-known fact:



Local defect:



Extended defect line:



Thus, geometric constraints can induce effective long-range interactions
(a well-known fact, actually).

As a result, thermodynamic limit may depend on boundary conditions.

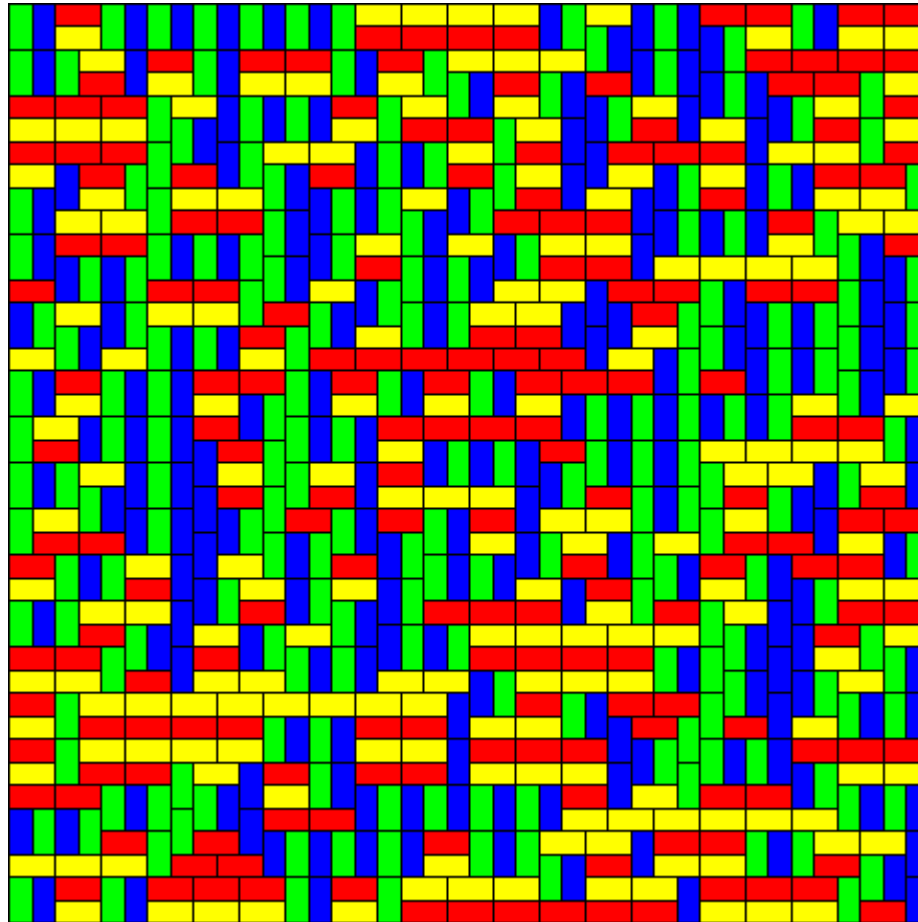
In particular, it may happen that:

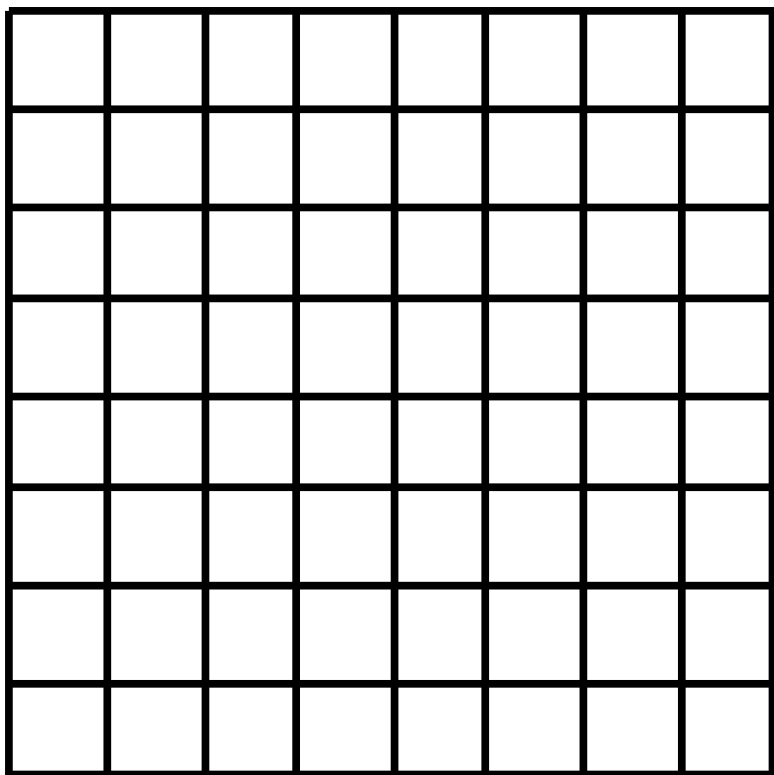
- order parameter acquires spatial dependence;
- free-energy and entropy densities acquire spatial dependence;
- spatial phase separation occurs, with emergence of regions of order and disorder sharply separated by some smooth curves.

Let us make this more definite:

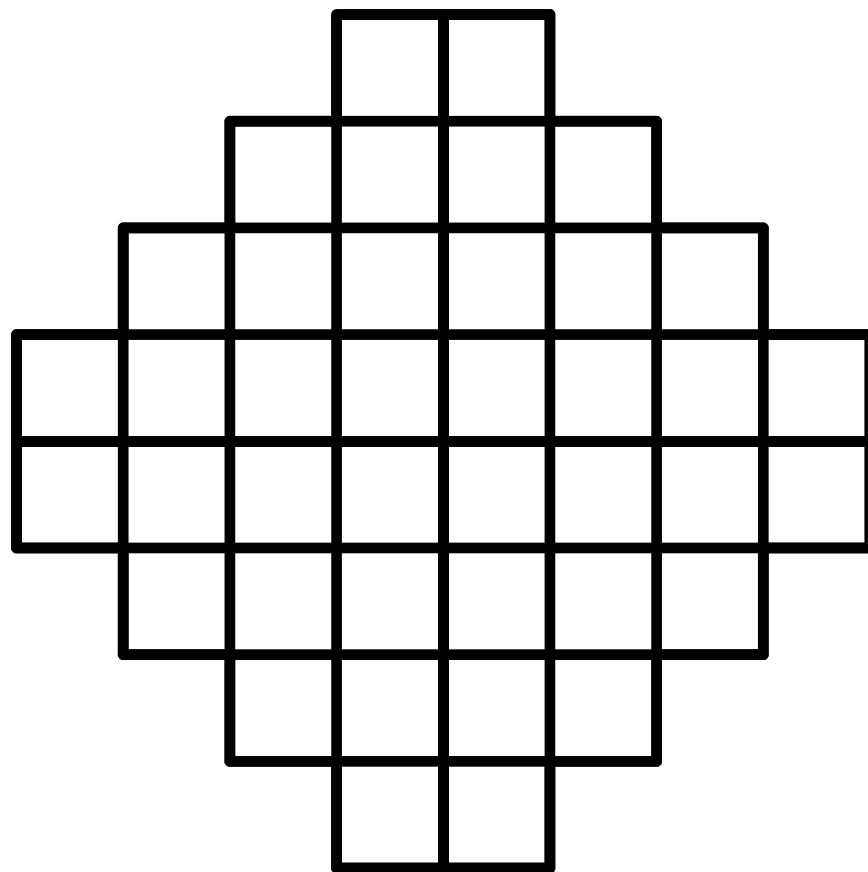
Domino tiling of a square

domino:= 2×1 tile





$N \times N$ Square

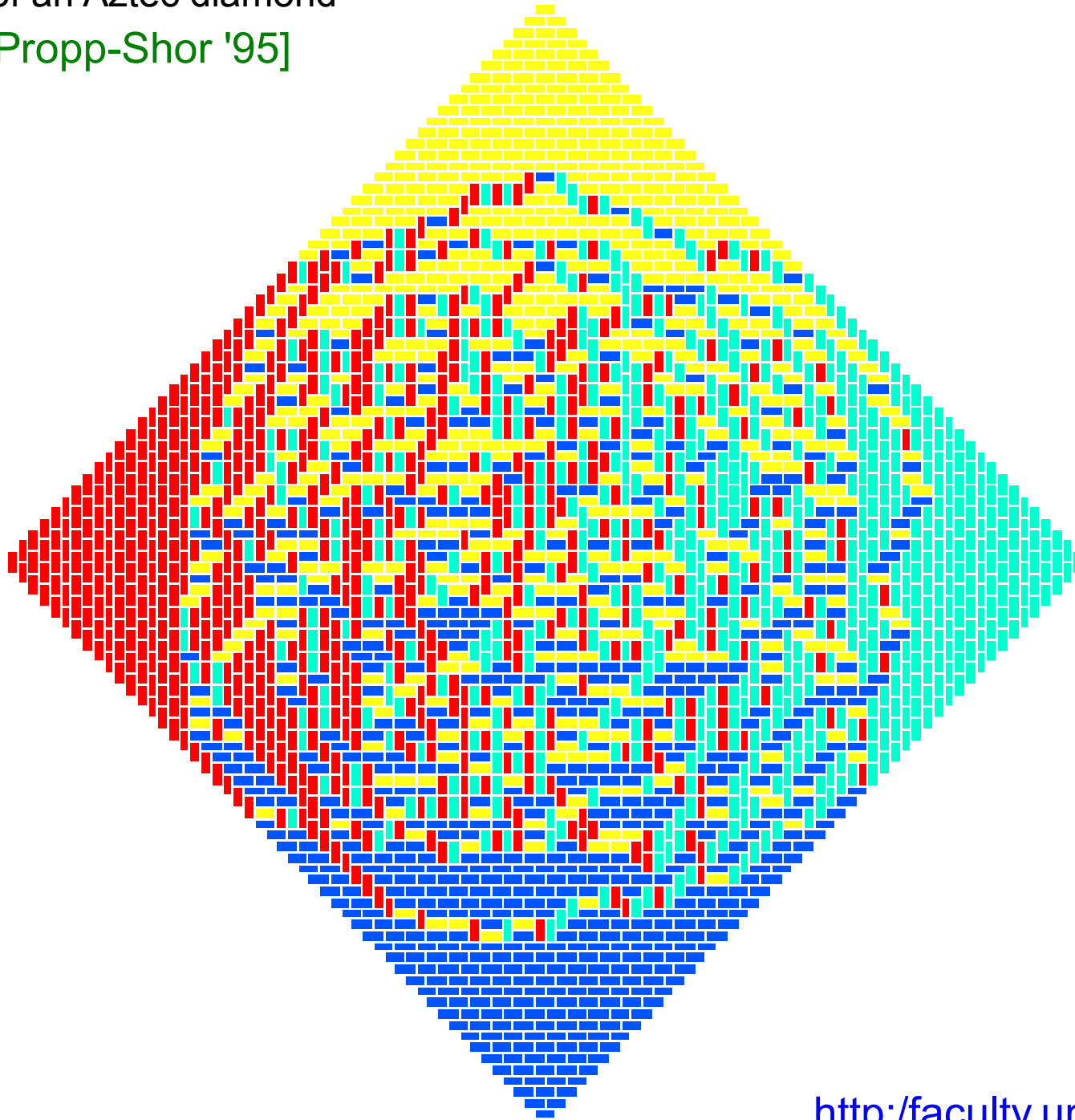


Aztec Diamond of Order N

($N = 8$)

Domino tiling of an Aztec diamond
[Jockush-Propp-Shor '95]

$N = 64$



The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

$\forall \epsilon > 0$, $\exists N$ such that “almost all” (i.e. with probability $P > 1 - \epsilon$) randomly picked domino tilings of $AD(N)$ have a temperate region whose boundary stays uniformly within distance ϵN from the circle of radius $N/\sqrt{2}$.

The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

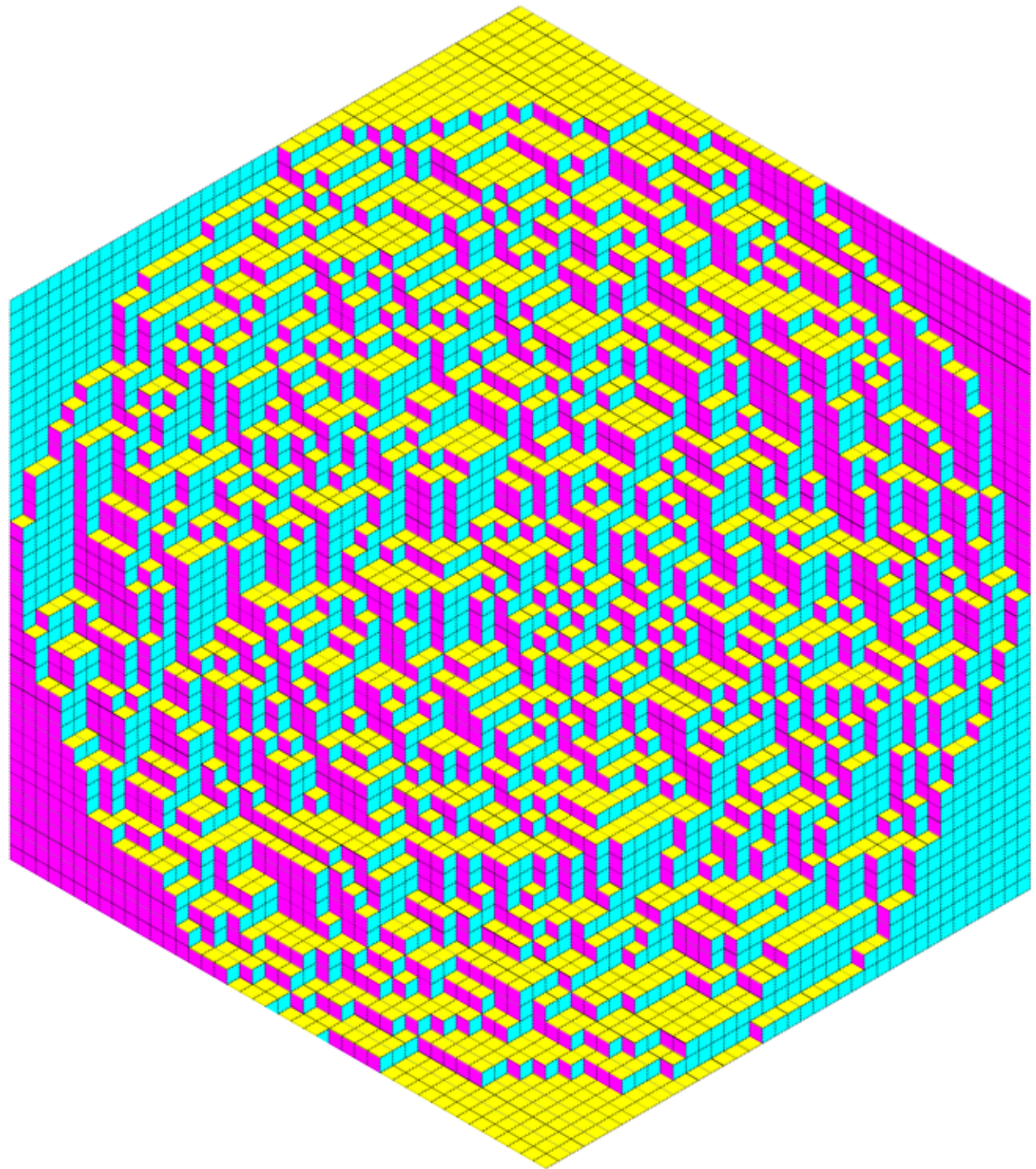
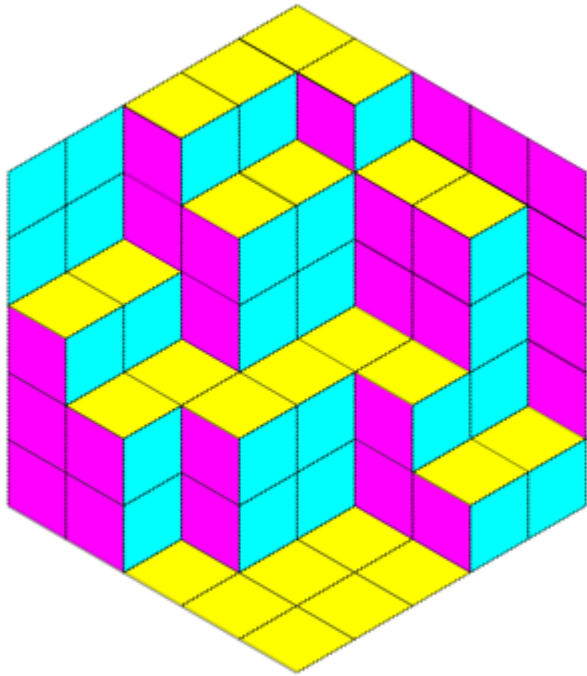
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Fluctuations:

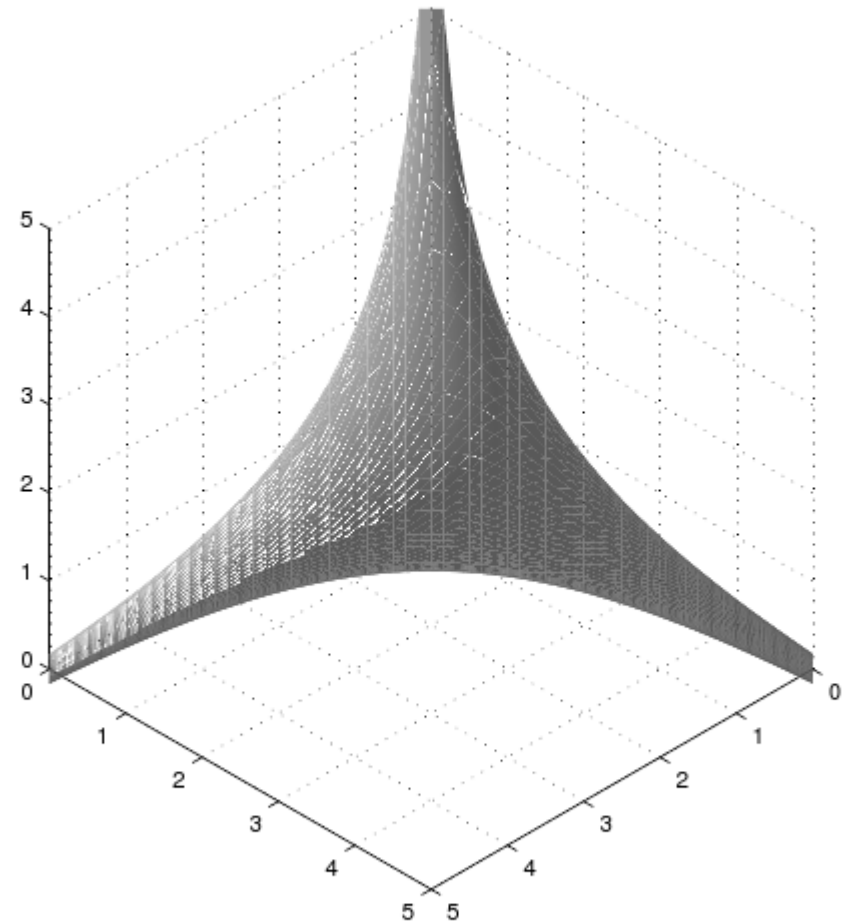
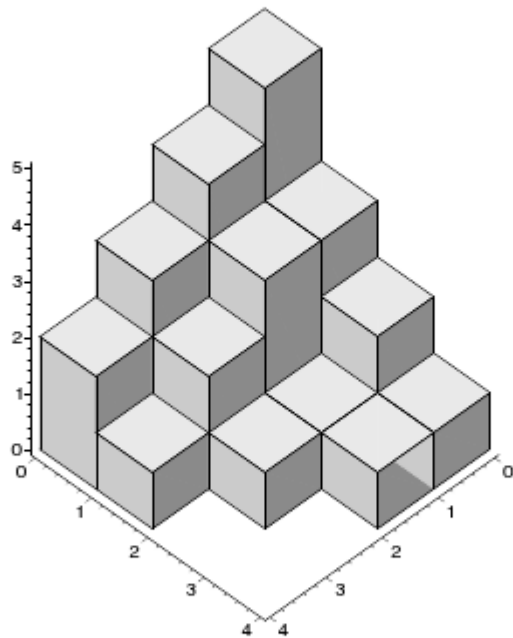
- boundary fluctuations $N^{1/3}$ [Johansson'00]
- fluctuations of boundary intersection with main diagonal obey Tracy-Widom distribution [Johansson'02]
- after suitable rescaling, boundary has limit as a random function, governed by an Airy stochastic process [Johansson'05]



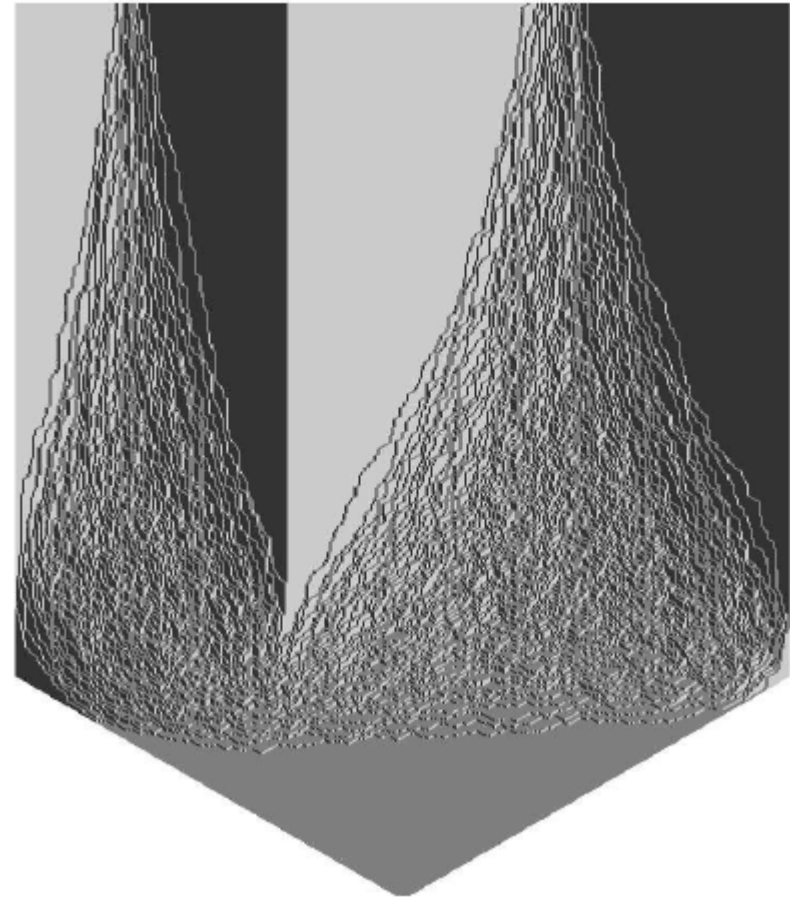
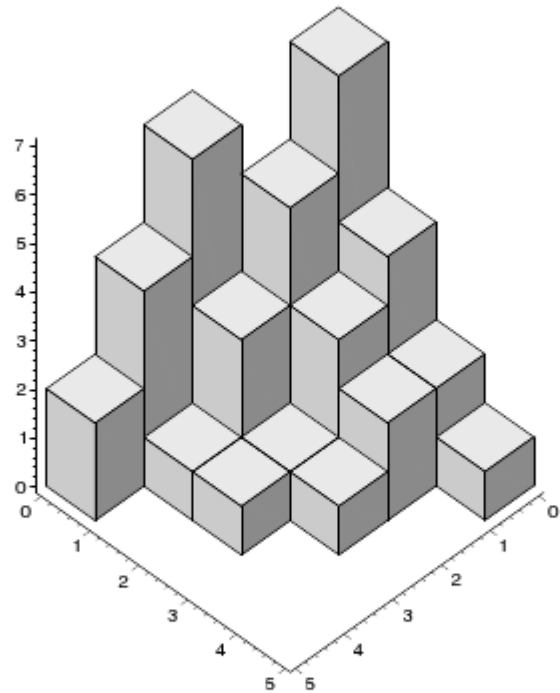
fluctuations are described by random matrix models



- Boxed plane partitions [Cohn-Larsen-Propp'98]



- Corner melting of a crystal [Ferrari-Spohn '02]
- Plane partitions [Cerf-Kenyon'01][Okounkov-Reshetikhin'01]

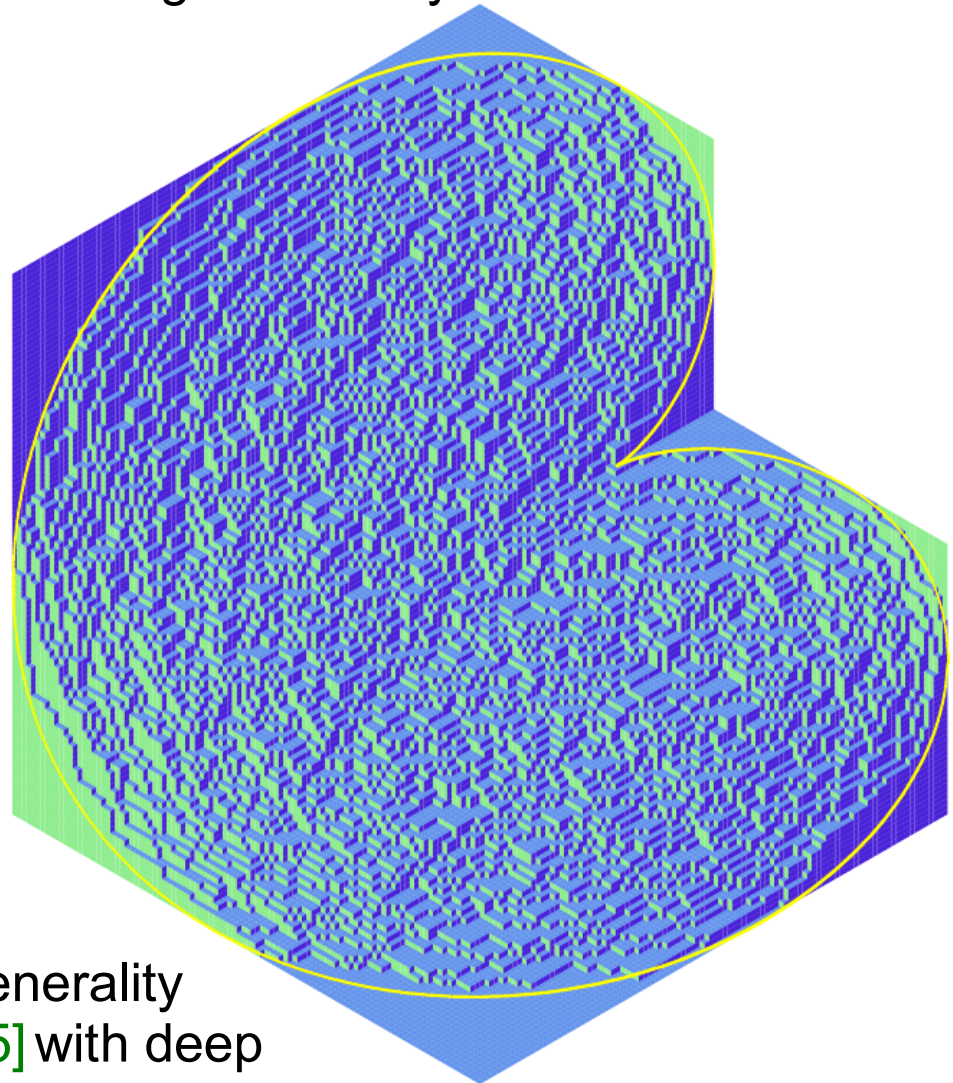


- Skewed plane partitions [Okounkov-Reshetikhin '05]
[Boutillier-Mkrtchyan-Reshetikhin-Tingley '10]

So we have seen:

- domino tilings;
- rhombi tilings;
- partitions; plane partitions; skewed plane partitions ...

Actually they are all avatars of the same model, '**dimer covering of regular planar bipartite lattices**', exhibiting emergence of phase separation, limit shapes, frozen boundaries /arctic curves, and fluctuations governed by Random Matrix models.

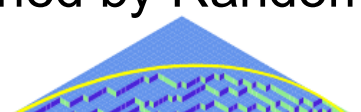


The model has been solved in full generality
[Kenyon, Sheffield, Okounkov, '03-'05] with deep
implications in algebraic geometry and algebraic combinatorics.

So we have seen:

- domino tilings;
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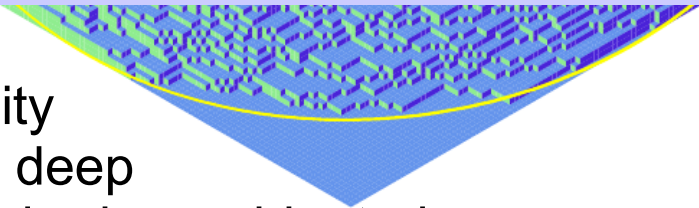
Actually they are all avatars of the same model, '**dimer covering of regular planar bipartite lattices**', exhibiting emergence of phase separation, limit shapes, frozen boundaries /arctic curves, and fluctuations governed by Random Matrix models.



Remark:

All these examples can be viewed as
2-d discrete free fermions

(or as dimer coverings with uniform probability)

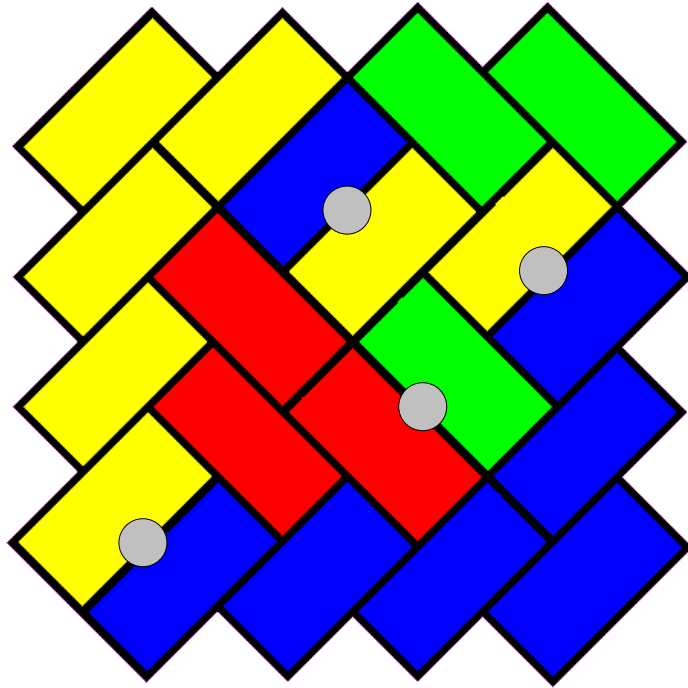


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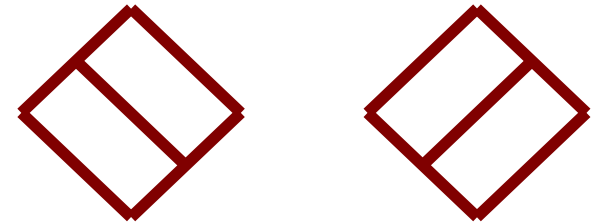
Introducing a non-uniform probability

(i.e., an interaction between dimers)

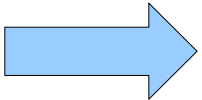
[Elkies-Kuperberg-Larsen-Propp'92]



Assign weight $c^2/2$ to:



and weights a and b , respectively, to:



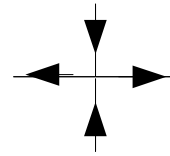
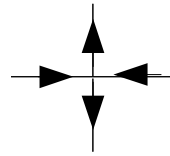
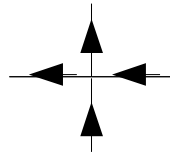
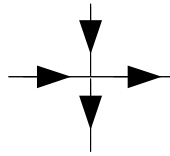
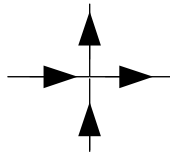
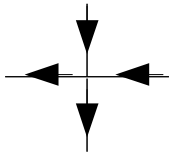
Six-vertex model with :

- free boundary conditions (square)
- Domain Wall b.c. (Aztec Diamond)

An exactly solvable model of statistical mechanics

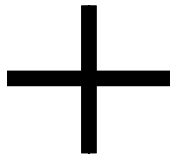
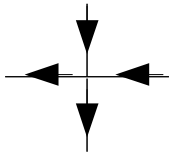
The six-vertex model

[Lieb '67] [Sutherland'67]

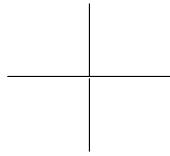
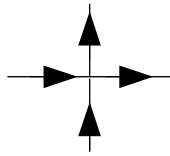


The six-vertex model

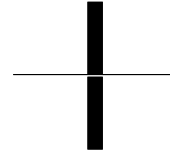
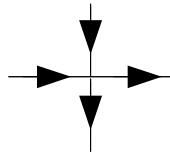
[Lieb '67] [Sutherland'67]



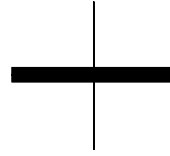
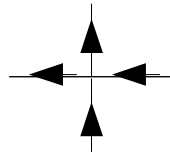
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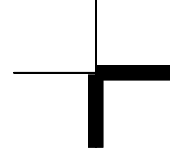
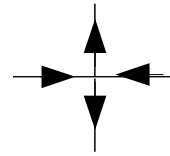
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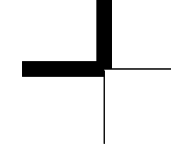
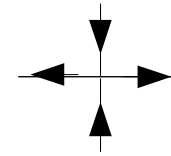
b



b



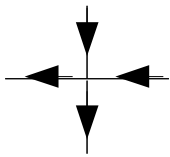
c



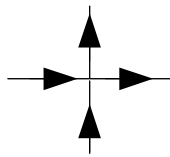
c

The six-vertex model

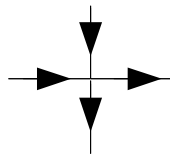
[Lieb '67] [Sutherland'67]



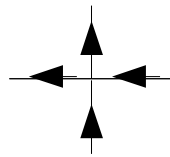
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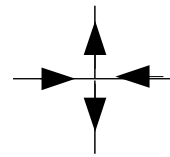
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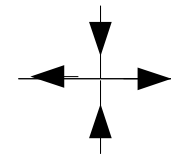
b



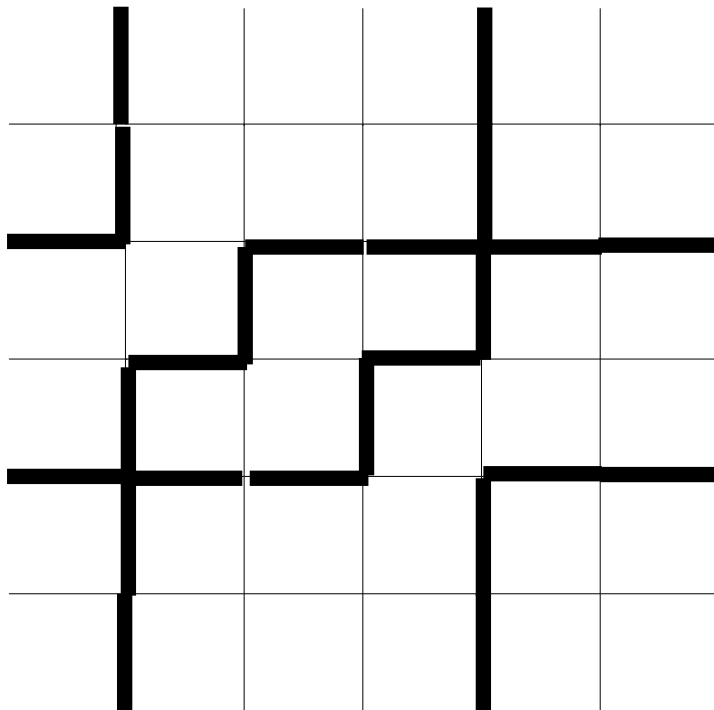
b



c



c



$$a = \sin(\lambda + \eta)$$

$$b = \sin(\lambda - \eta)$$

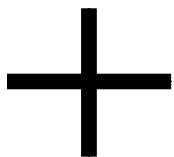
$$c = \sin 2\eta$$

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

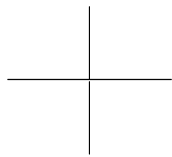
$$t = \frac{b}{a}$$

The Domain Wall six-vertex model

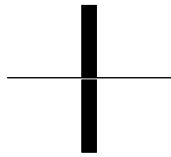
[Korepin '82]



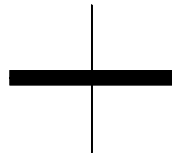
a



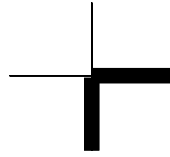
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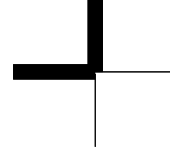
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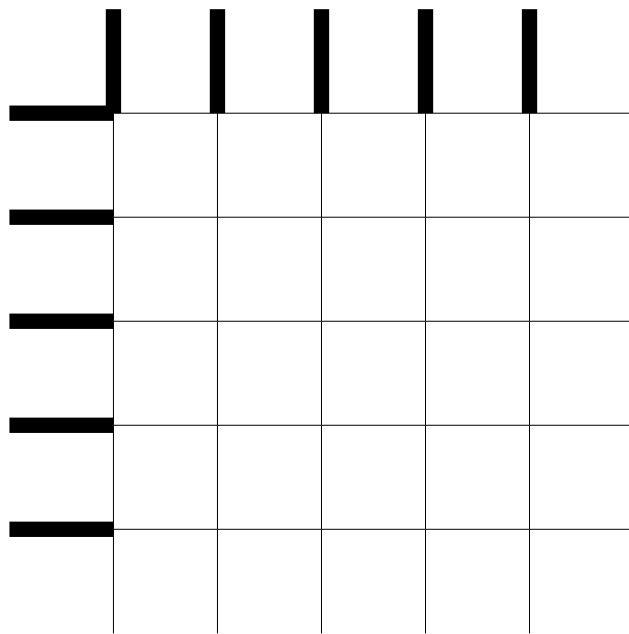
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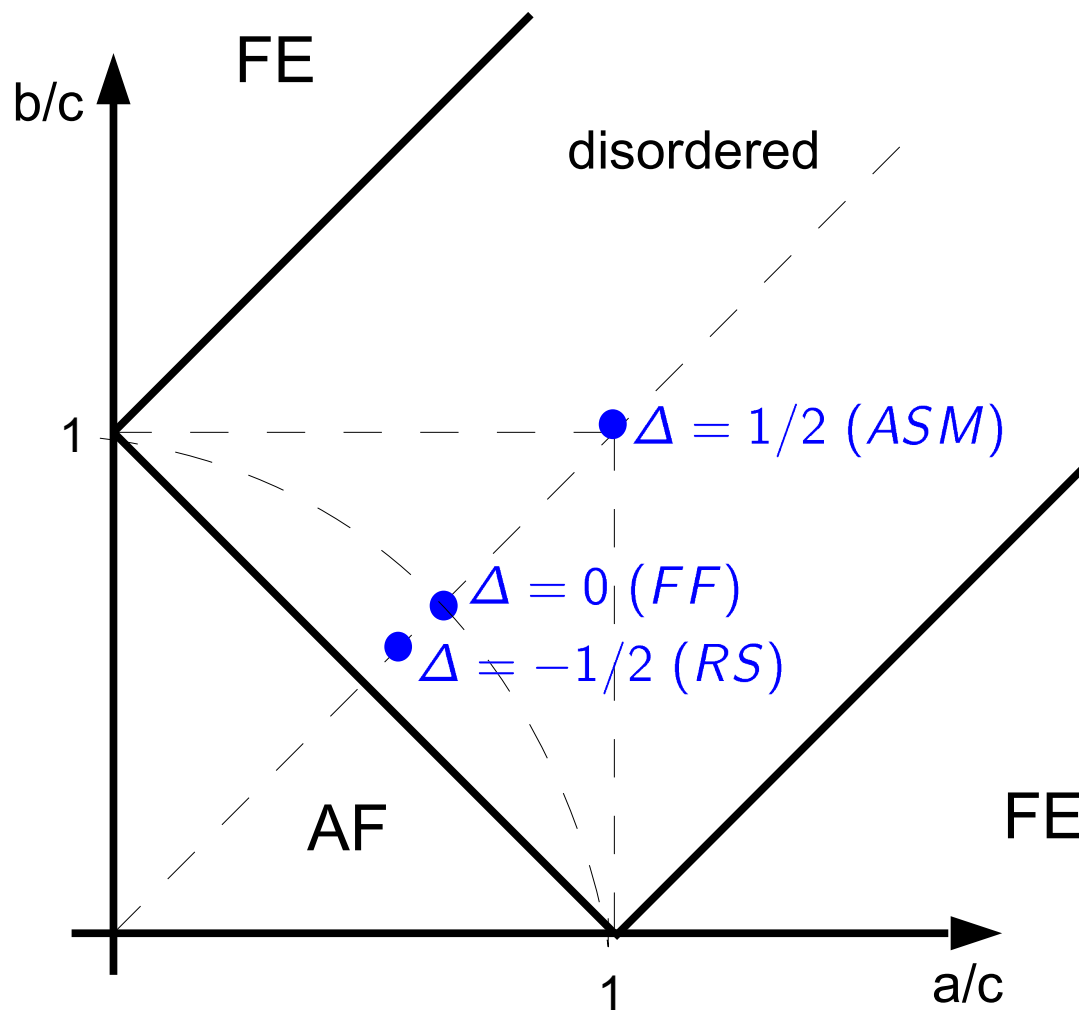


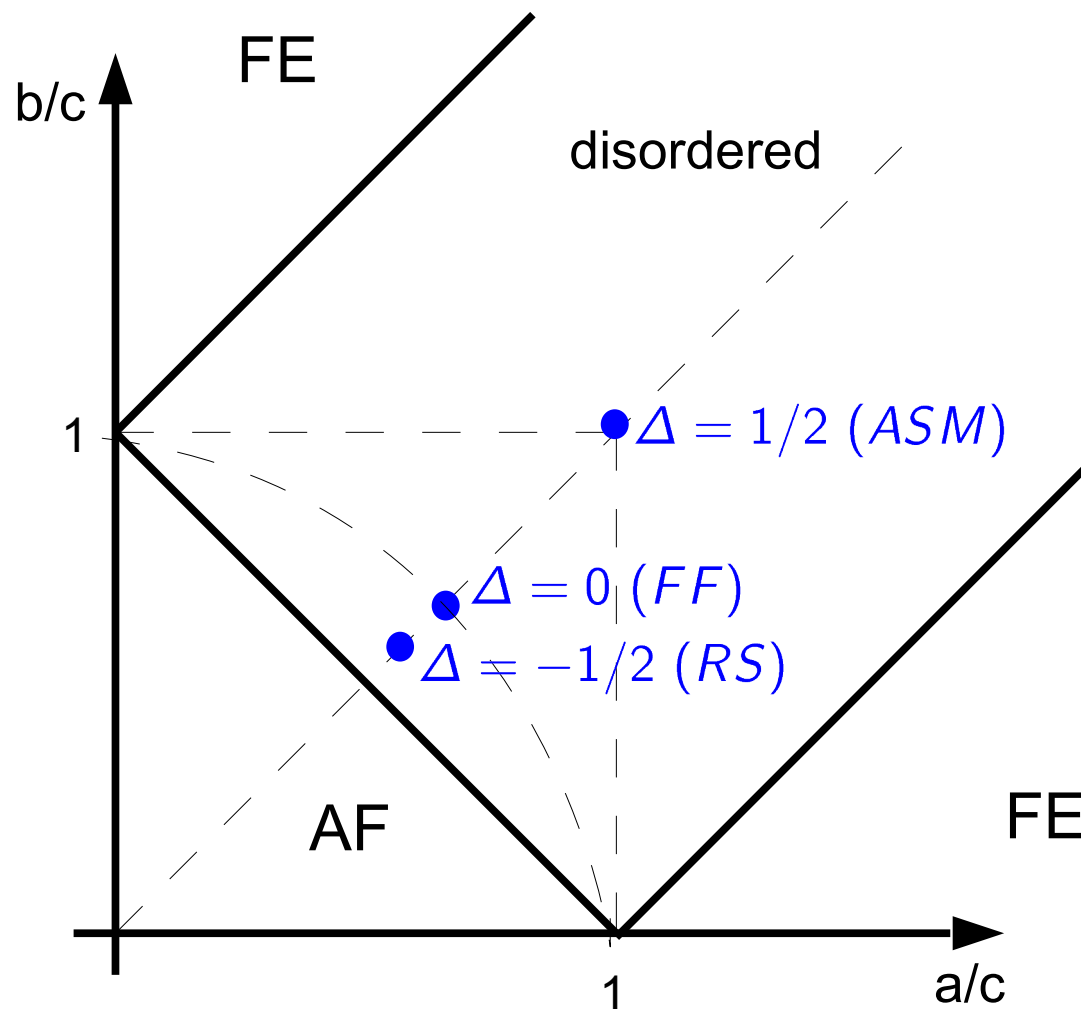
c



c





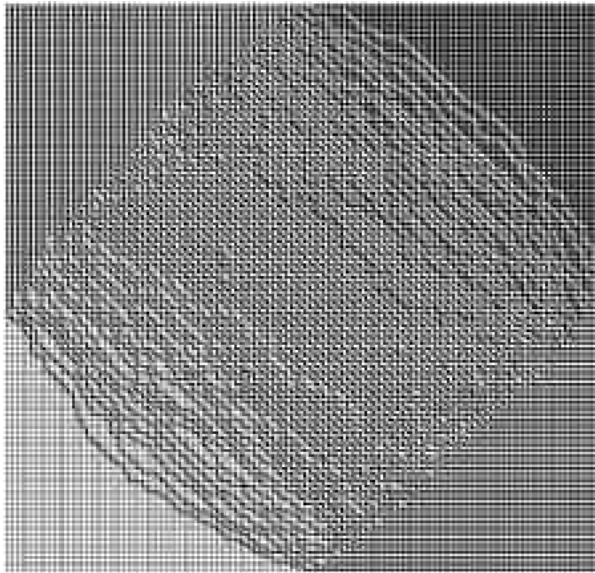


With Domain-Wall b.c., for $\Delta = 0$ we have the Arctic Circle Theorem.

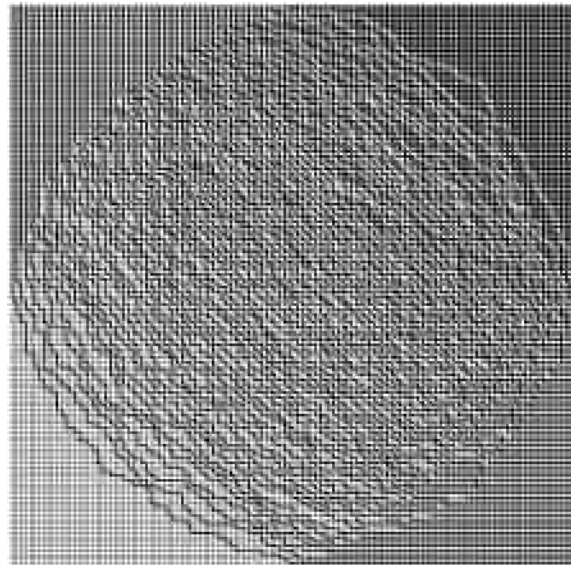
And for generic Δ and t ? And for generic regions? And what about fluctuations?

Domain Wall six-vertex model: numerical results

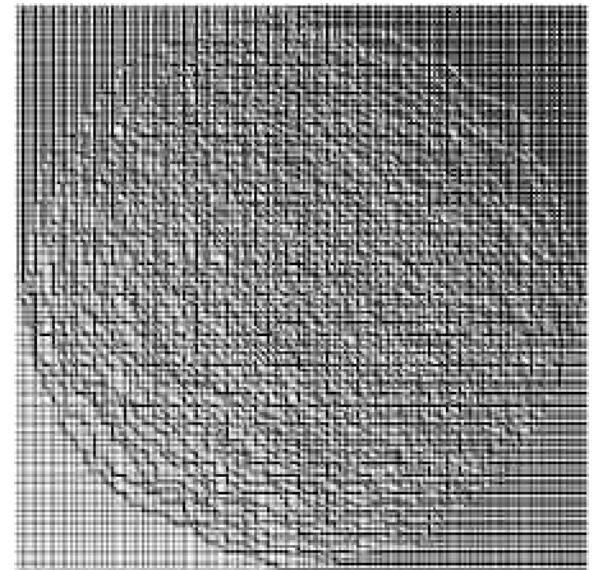
[Eloranta'99] [Zvonarev-Syluassen'04] [Allison-Reshetikhin'05]



$$\Delta = -3$$



$$\Delta = -0.92$$



$$\Delta = 0 \text{ (free fermions)}$$

$$N = 225$$

[Allison-Reshetikhin'05]

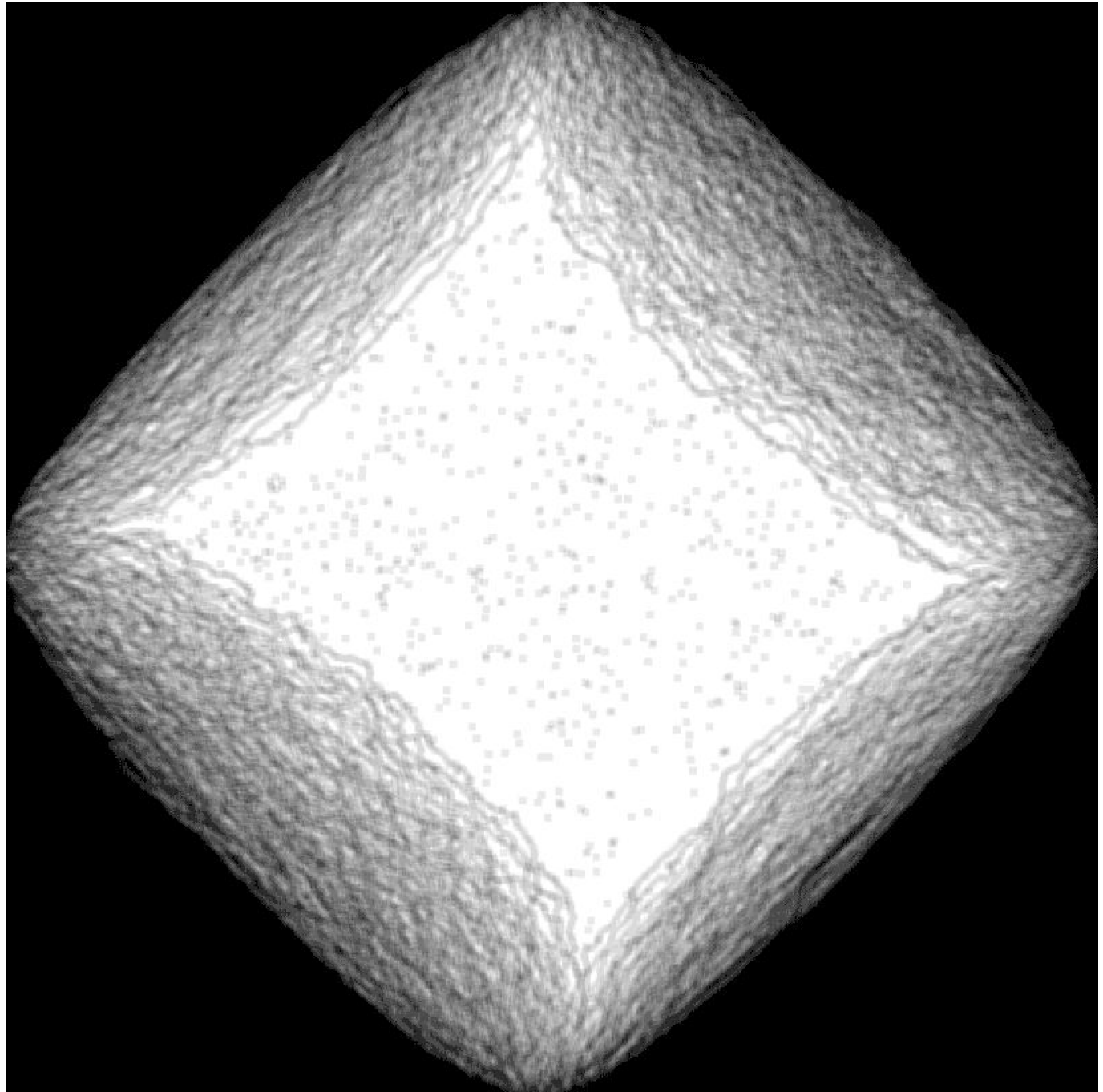
Area of disordered region increases with Δ .

(Anti-ferroelectric regime)

$$N = 1000$$

$$\Delta = -3$$

$$t = 0.5$$



White pixels represents c -vertices

[Allison-Reshetikhin'05]

Domain Wall six-vertex model: analytic results

(For generic Δ , not so many: translation invariance is broken!)

- Partition function:

- I-K determinant representation and Hankel determinant representation for Z_N [Korepin'82] [Izergin'87]
- Large N behaviour of Z_N :

Bulk free energy: DWBC $\neq$ PBC

[Korepin Zinn-Justin'00] [Zinn-Justin'01]

[Bleher-Fokin-Liechty'05-'09]

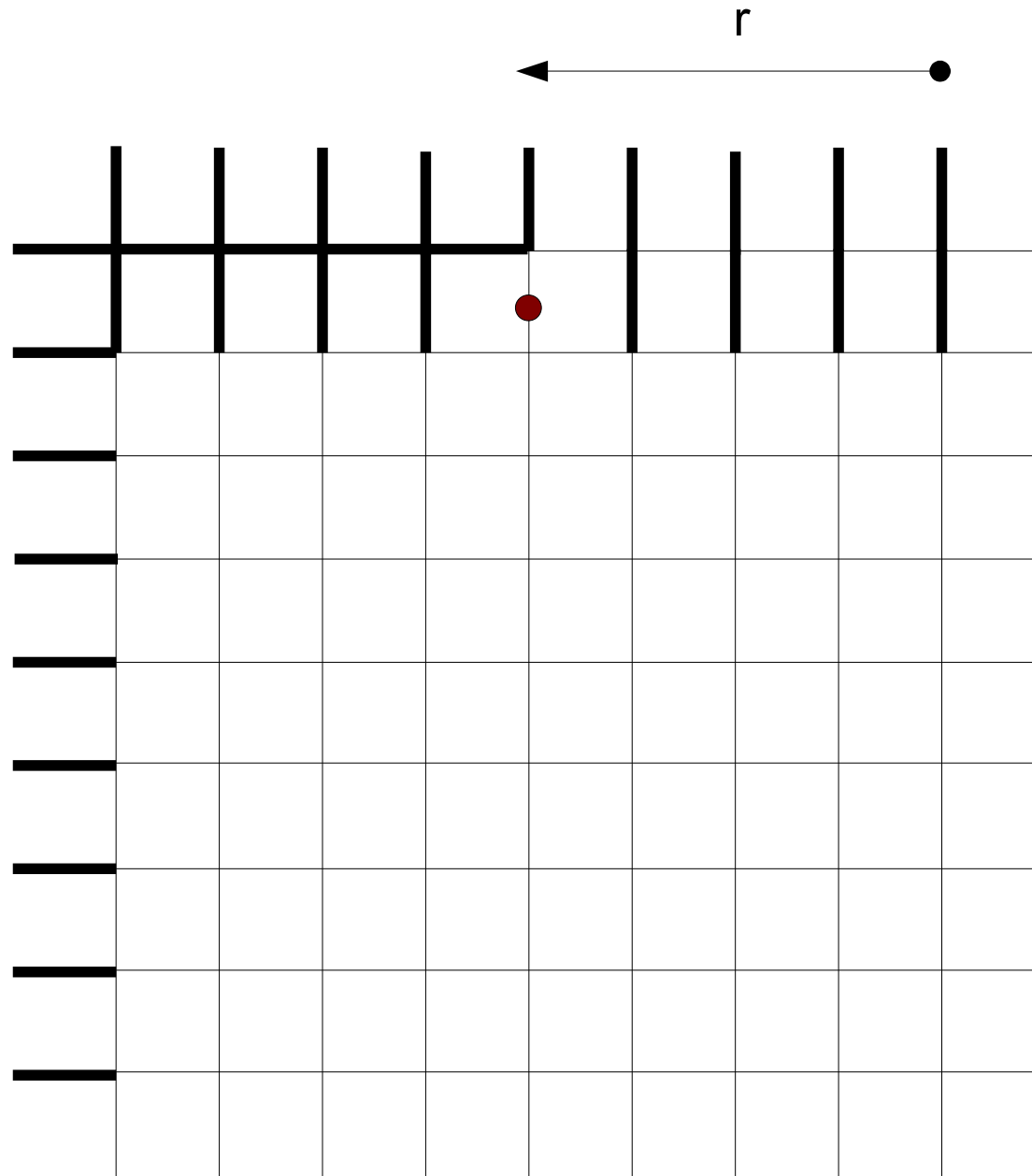
- Boundary correlation functions:

- one-point boundary correlation function

[Bogoliubov-Pronko-Zvonarev'02]

One-point boundary correlation function

$$H_N(r)$$



Domain Wall six-vertex model: analytic results

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[Korepin Zinn-Justin'00] [Zinn-Justin'01]

[Bleher-Fokin-Liechty'05-'09]

- Boundary correlation functions:

- one-point boundary correlation function
[Bogoliubov-Pronko-Zvonarev'02]
- two-point boundary correlation function [FC-Pronko'05]
(all these again in terms of $N \times N$ determinants)

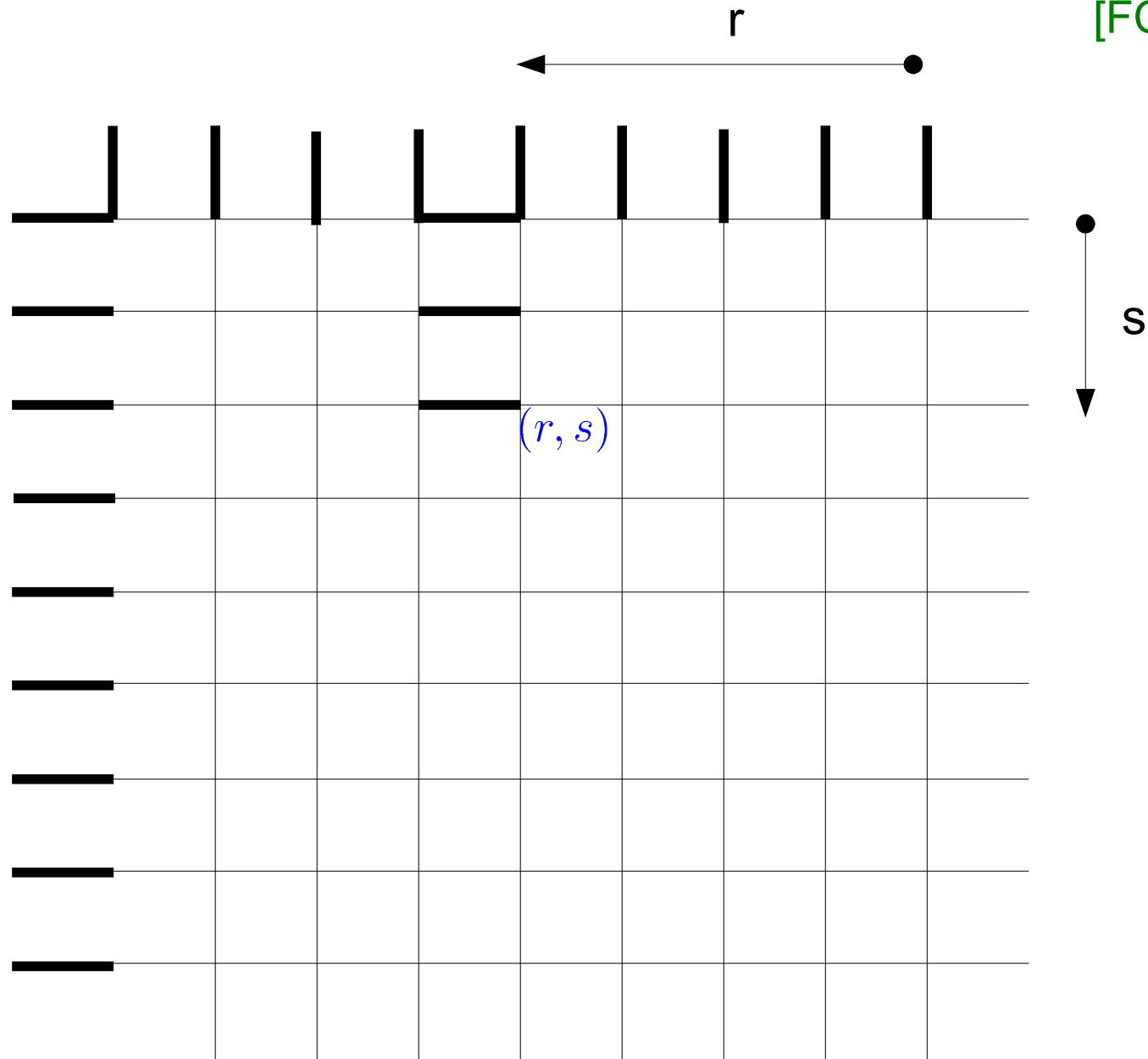
- Bulk correlation functions:

Nothing!

$F_N(r, s)$

Emptiness Formation Probability (EFP)

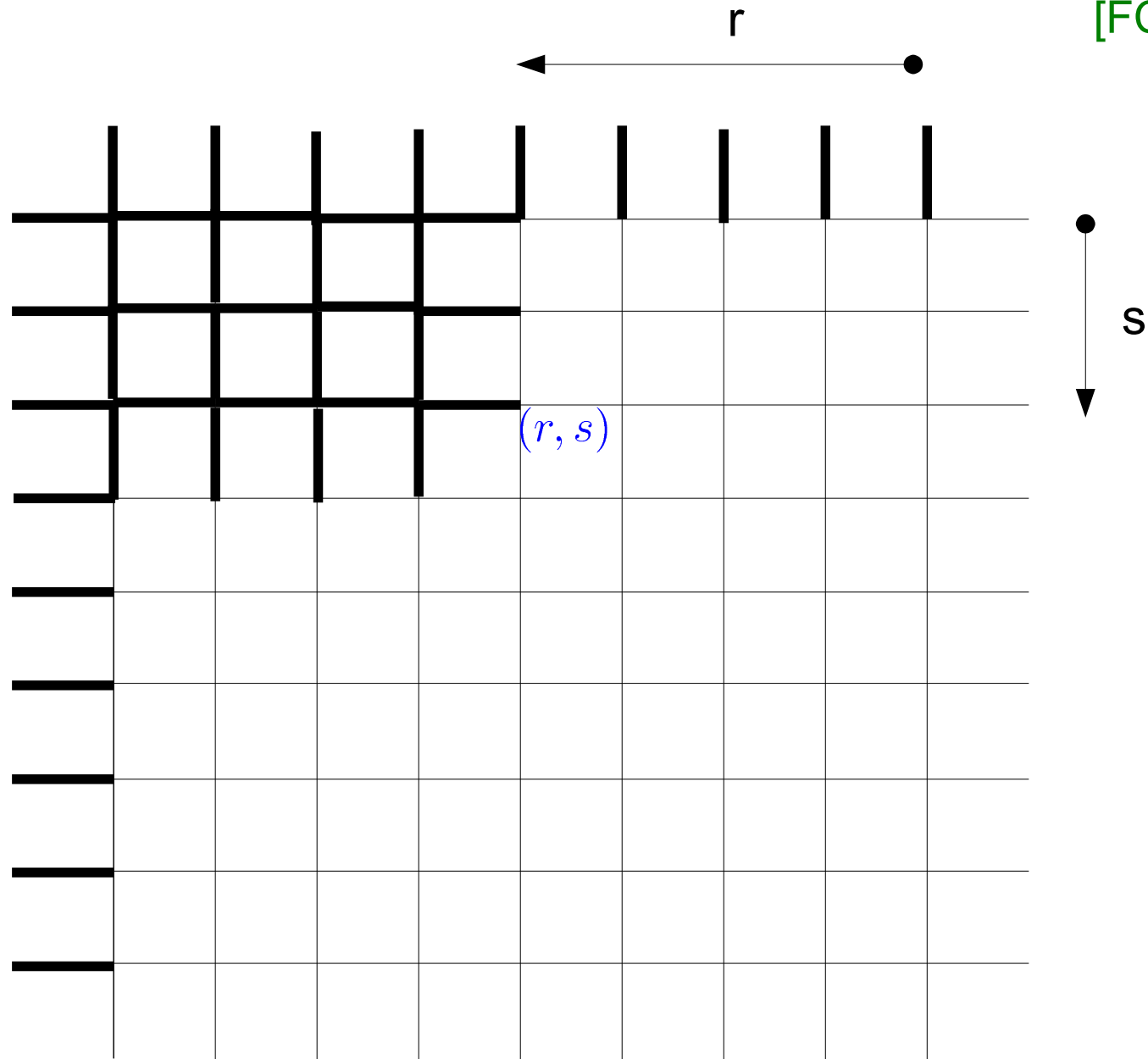
[FC-Pronko'08]



$$F_N(r, s)$$

Emptiness Formation Probability (EFP)

[FC-Pronko'08]



- Stepwise behaviour in correspondence of the Arctic curve
- Ability to discriminate only the top-left portion of the curve

Multiple Integral Representation for EFP

[FC-Pronko'08]

Define the generating function for the 1-point boundary correlator:

$$h_N(z) := \sum_{r=1}^N H_N(r) z^{r-1}, \quad h_N(1) = 1.$$

Now define, for $s = 1, \dots, N$:

$$h_N^{(s)}(z_1, \dots, z_s) := \frac{1}{\Delta_s(z_1, \dots, z_s)} \det_{1 \leq j, k \leq s} \left[h_{N-s+k}(z_j) (z_j - 1)^{k-1} z_j^{s-k} \right]$$

- The functions $h_N^{(s)}(z_1, \dots, z_s)$ are totally symmetric polynomials of order $N - 1$ in $z_1, \dots, z_s$.
- They define a new, alternative representation (with respect to Izergin-Korepin determinant) for the partition function Z_N .

Two important properties of $h_N^{(s)}(z_1, \dots, z_s)$:

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 0) = h_N(0) h_{N-1}^{(s-1)}(z_1, \dots, z_{s-1}),$$

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 1) = h_N^{(s-1)}(z_1, \dots, z_{s-1}).$$

Multiple Integral Representation for EFP

[FC-Pronko'08]

The following Multiple Integral Representation is valid ($r, s = 1, 2, \dots, N$):

$$F_N^{(r,s)} = \frac{(-1)^s Z_s}{s!(2\pi i)^s a^{s(s-1)} c^s} \oint_{C_0} \cdots \oint_{C_0} d^s z \prod_{j=1}^s \frac{[(t^2 - 2t\Delta)z_j + 1]^{s-1}}{z_j^r (z_j - 1)^s} \\ \times \prod_{\substack{j,k=1 \\ j \neq k}}^s \frac{z_k - z_j}{t^2 z_j z_k - 2t\Delta z_j + 1} h_{N,s}(z_1, \dots, z_s) h_{s,s}(u(z_1), \dots, u(z_s))$$

where $u(z) := -\frac{z-1}{(t^2-2t\Delta)z+1}$.

Ingredients:

- Quantum Inverse Scattering Method to obtain a recurrence relation, which is solved in terms of a determinant representation on the lines of Izergin-Korepin formula;
- Orthogonal Polynomial and Random Matrices technologies to rewrite it as a multiple integral.

Remark:

Similar expressions occurs for correlation function in ASEP [Tracy-Widom'08-'11].

Scaling limit of EFP

[FC-Pronko'10]

Evaluate:

$$F(x, y) := \lim_{N \rightarrow \infty} F_N(xN, yN) \quad x, y \in [0, 1]$$

in the limit:

$$N, r, s \rightarrow \infty$$

$$\frac{r}{N} = x$$

$$\frac{s}{N} = y$$

using Saddle-Point method.

NB1:

- $s \times s$ Vandermonde determinant
- s -order pole at $z = 1$



Penner Random Matrix model
[Penner'88]

NB2:

- By construction, in the scaling limit, EFP is 1 in the frozen region, and 0 in the disordered one, with a stepwise behaviour in correspondence of the Arctic curve.
- From the structure of the Multiple Integral Representation, such stepwise behaviour can be ascribed to the position of the SPE roots with respect to the pole at $z = 1$.
- The considered generalized Penner model allows for condensation of 'almost all' SPE roots at $z = 1$. [Tan'92] [Ambjorn-Kristjansen-Makeenko'94]

Condensation of 'almost all'
SPE roots at $z = 1$



Arctic Curves

Mathematically, the condition of total condensation (i.e. the Arctic curve) is given by:

$$\frac{y}{z-1} - \frac{x}{z} - \frac{yt^2}{t^2z - 2\Delta t + 1} + \lim_{N \rightarrow \infty} \frac{1}{N} \partial_z \ln h_N(z) = 0$$

must have two coinciding real roots in interval: $z \in [1, \infty]$.

Remarks:

- Arctic curve depends ONLY on the BOUNDARY correlation function!
(a miracle of integrability)
- We thus need to evaluate $\lim_{N \rightarrow \infty} \frac{1}{N} \partial_z \ln h_N(z)$ for generic Δ, t :
disordered regime, $|\Delta| < 1$ [FC-Pronko'10];
anti-ferroelectric regime, $\Delta < -1$ [FC-Pronko-Zinn-Justin'10].
- It appears that the Arctic curve is: algebraic, in 'roots of unity' cases;
transcendent, otherwise.
[FC-Pronko'10],[FC-Pronko-Noferini'11]
- This differs from the dimer covering case, where only algebraic curves appear, for any choice of lattice and b.c. [Kenyon-Okounkov-Sheffield'03-'05].

Evaluation of $h_N(z)$ (disordered regime $|\Delta| < 1$)

[FC-Pronko'10]

For $|\Delta| < 1$, we have

$$h_N(z(\zeta)) \underset{N \rightarrow \infty}{\sim} \left[\frac{\sin \gamma(\lambda - \eta)}{\gamma \sin(\lambda - \eta)} \right]^N \left[\frac{\sin(\zeta + \lambda - \eta) \sin(\gamma \zeta)}{\sin \gamma(\zeta + \lambda - \eta) \sin \zeta} \right]^N e^{o(N)}$$

where

$$z(\zeta) = \frac{\sin(\lambda + \eta)}{\sin(\lambda - \eta)} \frac{\sin(\zeta + \lambda - \eta)}{\sin(\zeta + \lambda + \eta)}, \quad \text{and} \quad \gamma := \frac{\pi}{\pi - 2\eta}.$$

$$\Delta = \cos 2\eta \quad t = \frac{\sin(\lambda - \eta)}{\sin(\lambda + \eta)}$$

NB: $z \in [1, +\infty)$ corresponds to $\zeta \in [0, \pi - \lambda - \eta)$

Evaluation of $h_N(z)$ (anti-ferroelectric regime $|\Delta| < -1$)

[FC-Pronko-Zinn-Justin'10]

For $\Delta < -1$, the large N behaviour of $h_N(z)$ is given by

$$h_N(z) \underset{N \rightarrow \infty}{\sim} \left[\frac{\vartheta_1(\gamma(\lambda + \eta))}{\gamma \sinh(\lambda + \eta)} \right]^N \left[\frac{\sinh(\zeta + \lambda + \eta) \vartheta_1(\gamma \zeta)}{\vartheta_1(\gamma(\zeta + \lambda + \eta)) \sinh \zeta} \right]^N e^{o(N)}$$

where Jacobi Theta function ϑ_1 has nome $q = e^{\pi^2/(2\eta)}$.

We have

$$z(\zeta) = -\frac{\sinh(\eta - \lambda) \sinh(\eta + \lambda + \zeta)}{\sinh(\eta + \lambda) \sinh(\eta - \lambda - \zeta)}, \quad \text{and} \quad \gamma := \frac{\pi}{2\eta}.$$

$$\Delta = -\cosh 2\eta \quad t = \frac{\sinh(\eta + \lambda)}{\sinh(\eta - \lambda)}$$

NB: $z \in [1, +\infty)$ corresponds to $\zeta \in [0, \lambda + \eta)$

In both cases we get the Arctic curve in parametric form ($\zeta \in [0, \zeta_{max}]$):

$$\begin{aligned}
 x &= \frac{1}{\Phi(\zeta + \lambda - \eta, 2\eta)\Psi(\zeta, 2\eta) - \Psi(\zeta + \lambda - \eta, 2\eta)\Phi(\zeta, 2\eta)} \\
 &\quad \times \left\{ \left[\Psi(\zeta, \lambda - \eta) - \gamma^2 \Psi(\gamma\zeta, \gamma(\lambda - \eta)) \right] \Phi(\zeta, 2\eta) \right. \\
 &\quad \left. - \left[\Phi(\zeta, \lambda - \eta) - \gamma \Phi(\gamma\zeta, \gamma(\lambda - \eta)) \right] \Psi(\zeta, 2\eta) \right\} , \\
 y &= \frac{1}{\Phi(\zeta + \lambda - \eta, 2\eta)\Psi(\zeta, 2\eta) - \Psi(\zeta + \lambda - \eta, 2\eta)\Phi(\zeta, 2\eta)} \\
 &\quad \times \left\{ \left[\Psi(\zeta, \lambda - \eta) - \gamma^2 \Psi(\gamma\zeta, \gamma(\lambda - \eta)) \right] \Phi(\zeta + \lambda - \eta, 2\eta) \right. \\
 &\quad \left. - \left[\Phi(\zeta, \lambda - \eta) - \gamma \Phi(\gamma\zeta, \gamma(\lambda - \eta)) \right] \Psi(\zeta + \lambda - \eta, 2\eta) \right\} .
 \end{aligned}$$

where

$$\gamma := \frac{\pi}{\pi - \arccos \Delta}$$

$$\Phi(\mu) := \frac{\sin(2\eta)}{\sin(\mu + \eta) \sin(\mu - \eta)} ,$$

$$\Psi(\zeta) := \cot \zeta - \cot(\zeta + \lambda - \eta) - \gamma \cot \gamma\zeta + \gamma \cot \gamma(\zeta + \lambda - \eta) ,$$

(Disordered regime)

or

$$(-1 < \Delta < 1)$$

$$\Phi(\mu) := \frac{\sinh(2\eta)}{\sinh(\eta - \mu) \sinh(\eta + \mu)} ,$$

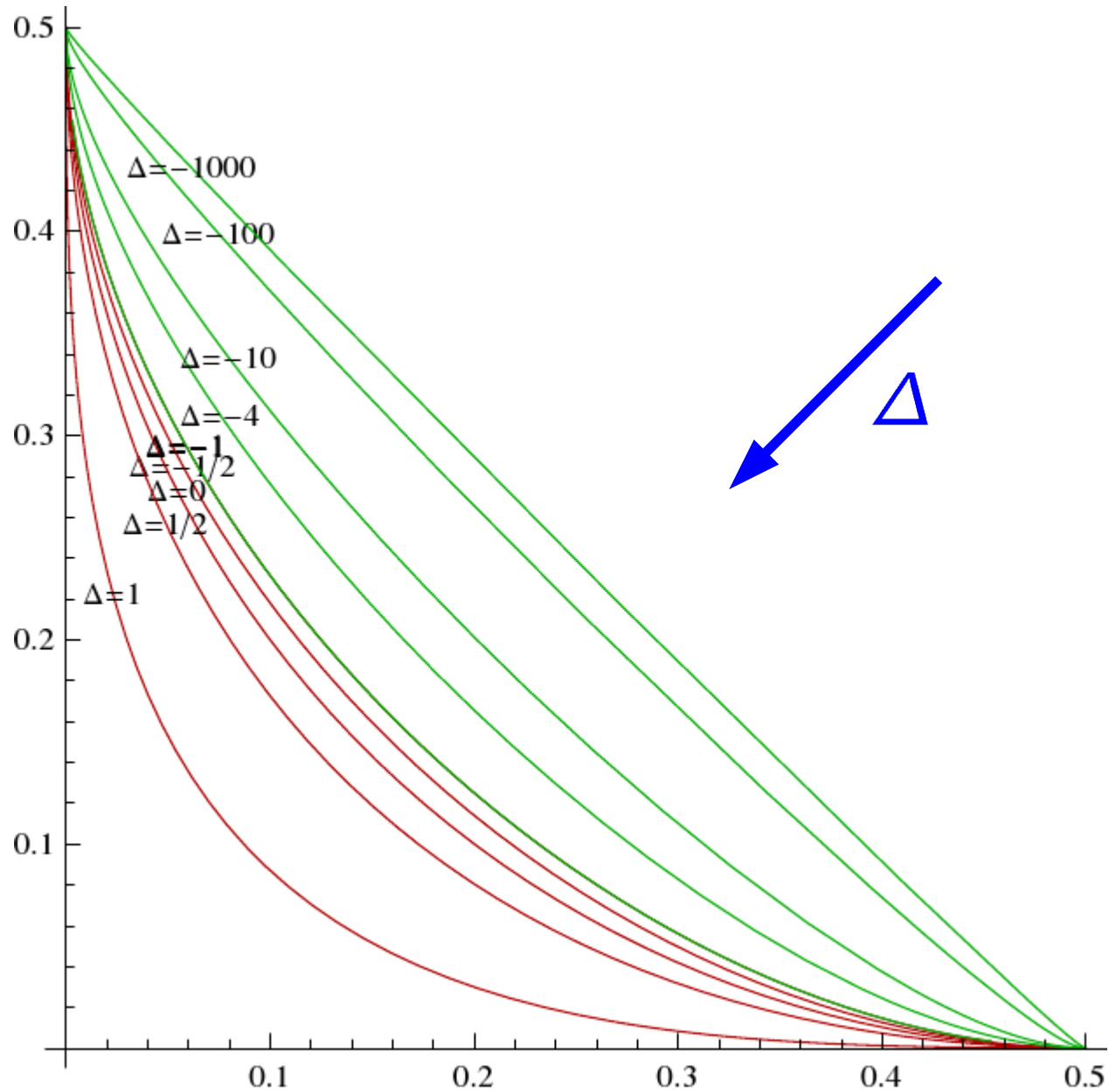
$$\Psi(\zeta) := \cot \zeta - \coth(\eta - \lambda - \zeta) - \gamma \frac{\vartheta_1'(\gamma\zeta)}{\vartheta_1(\gamma\zeta)} + \gamma \frac{\vartheta_1'(\gamma(\zeta + \lambda - \eta))}{\vartheta_1(\gamma(\zeta + \lambda - \eta))} ,$$

(Anti-ferroelectric regime)

$$(-\infty < \Delta \leq -1)$$

NB: $(\Delta, t, z) \longrightarrow (\eta, \lambda, \zeta)$

Red curve: disordered regime [FC-Pronko'10]
 Green curve: anti-ferroelectric regime [FC-Pronko-Zinn-Justin'10]



NB: $\Delta = 0 \implies$ Arctic Circle
 $\Delta = -\infty \implies$ straight line

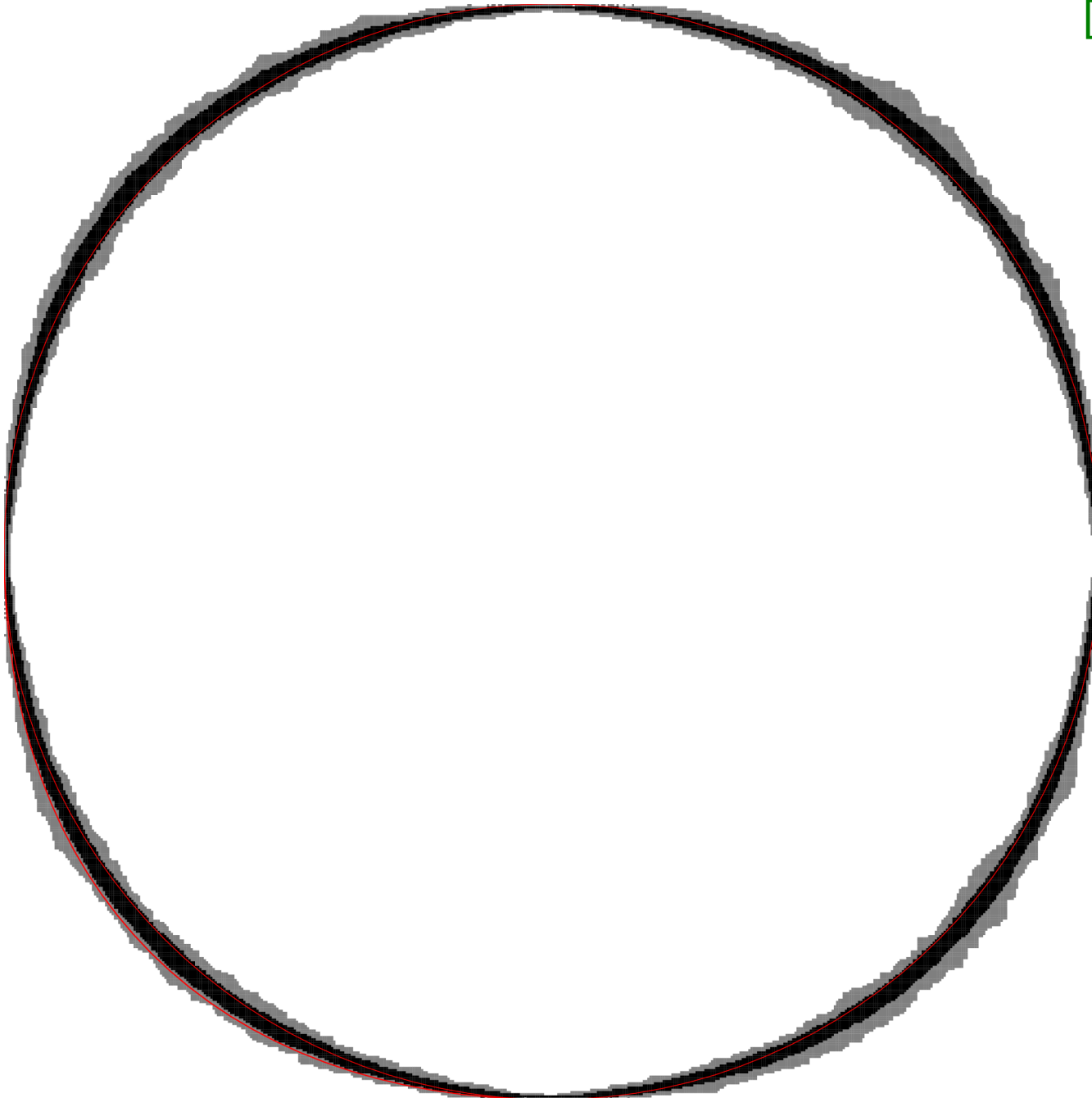
$$(2x - 1)^2 + (2y - 1)^2 - 4xy = 1, \quad x, y \in [0, \tfrac{1}{2}].$$

[FC-Pronko'10]

ASMs: N=500

199 samples

$$\Delta = 1/2$$

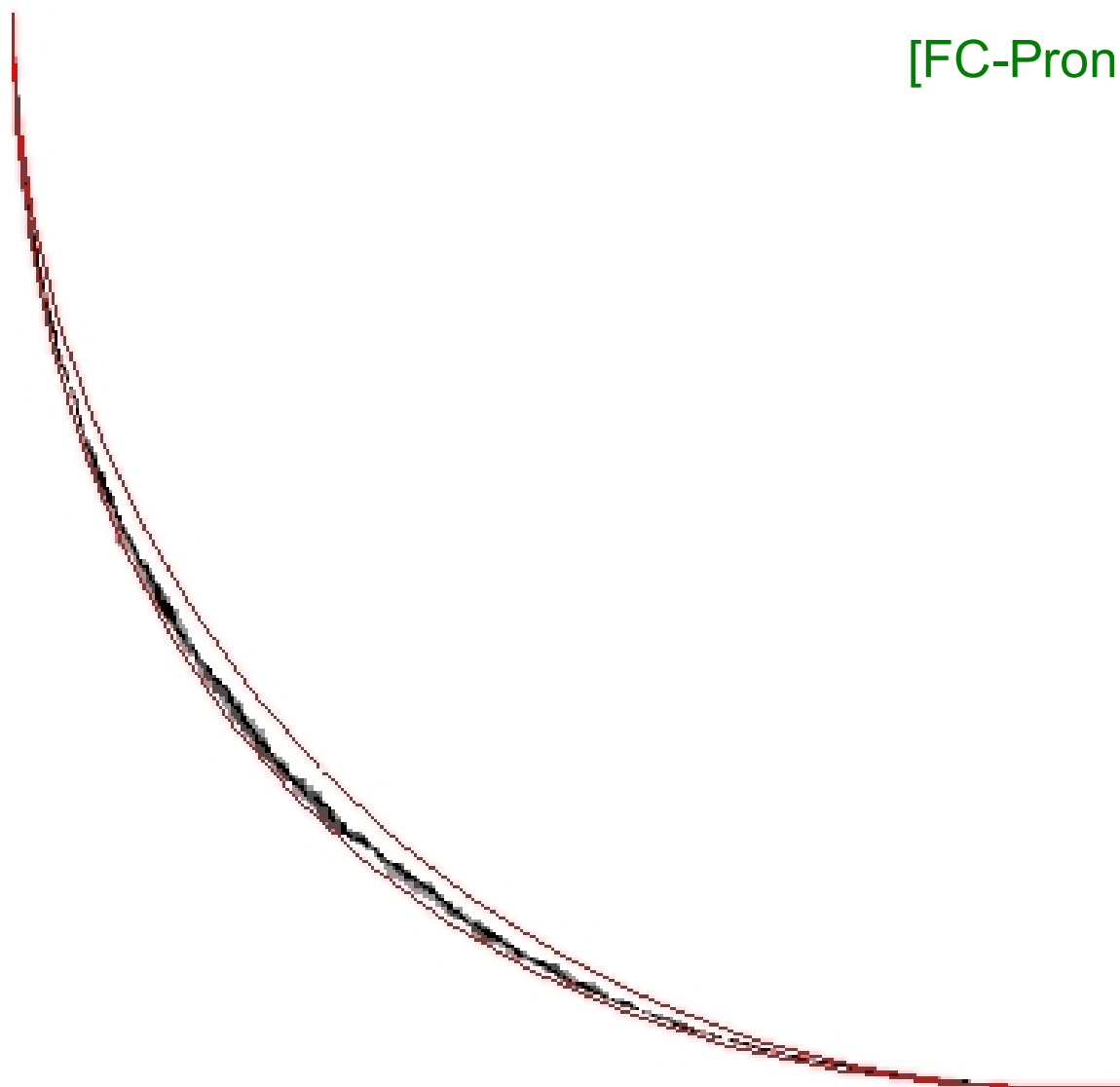


ASMs: $N=1500$

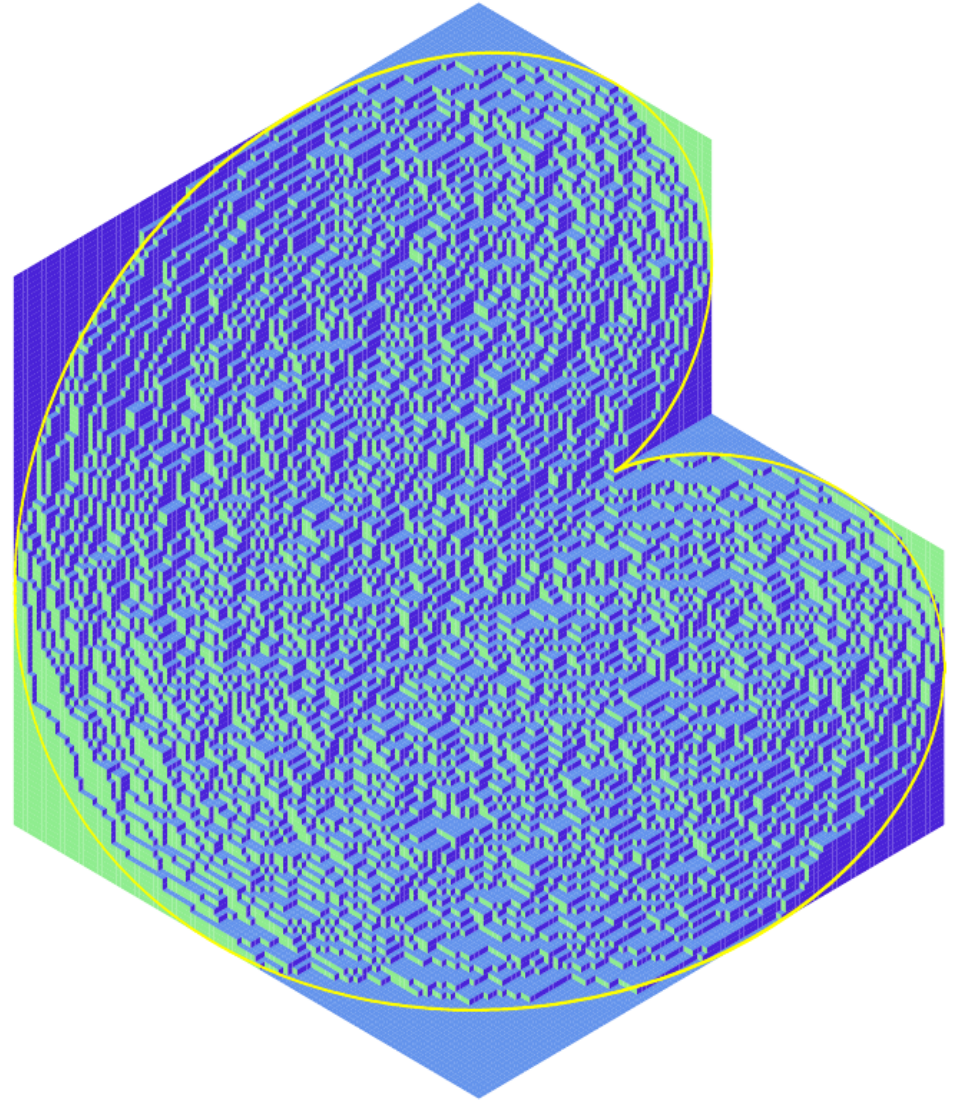
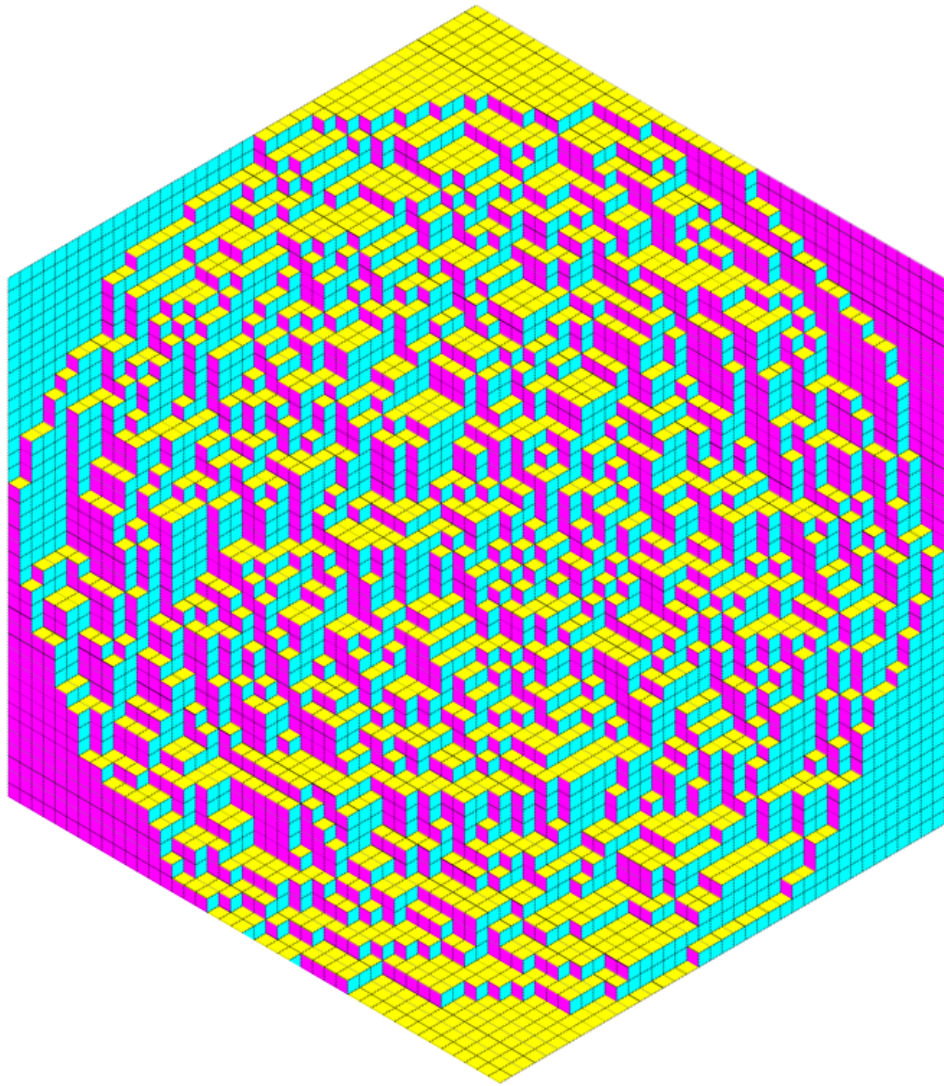
10 samples

$$\Delta = 1/2$$

[FC-Pronko'10]



Tilings of generic regions?



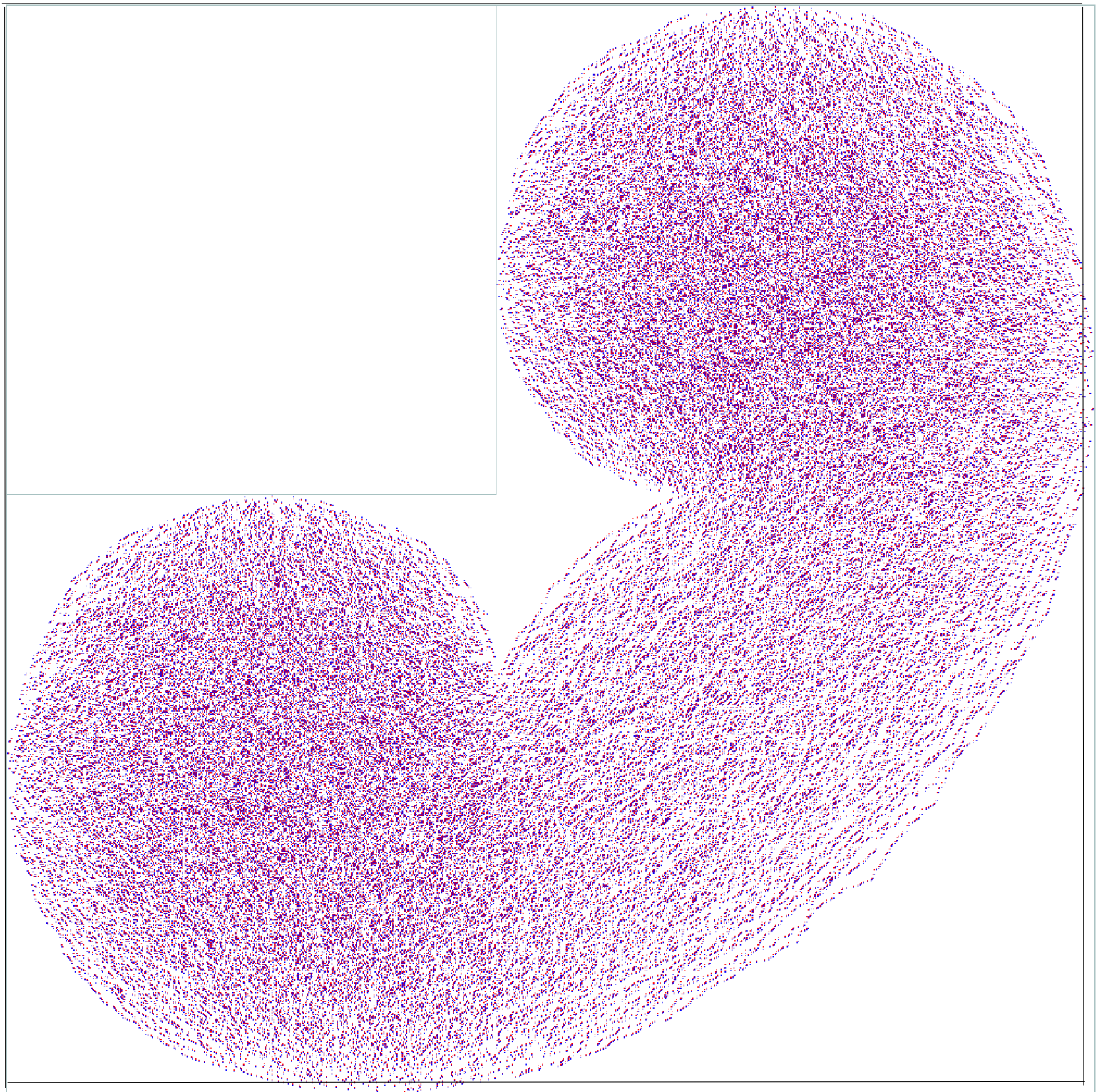
Use the theory provided by [Kenyon-Okounkov-Sheffield'03] [Kenyon-Okounkov'03,'05]

6VM

$$\Delta = \frac{1}{2}$$

$$N = 1000$$

$$r, s = 450$$



Six-vertex model with generic (fixed) BC?

[FC-Sportiello, *in progress*]

Our previous result on the Arctic curve in a square domain can be rephrased as follows:

The arctic curve is the geometric caustic (envelope) of the family of straight lines:

$$-x\frac{1}{z} + y\frac{(t^2 - 2\Delta t + 1)}{(z - 1)(t^2 z - 2\Delta t + 1)} + \lim_{N \rightarrow \infty} \frac{1}{N} \partial_z \ln h_N(z) = 0, \quad z \in [1, +\infty)$$

Questions:

- What is the geometrical meaning of this family of straight line?
 - why the constant term is determined by the boundary correlator $h_N(z)$?
 - what determines the angular coefficient of these lines?

Understanding this would provide:

- an alternative (geometrical) derivation of the Arctic curve;
- a geometrical strategy to attack the problem of Arctic curves in generic domains.

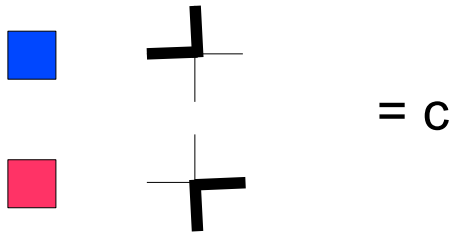
Some numerical results

[FC-Sportiello, *in progress*]

From now on we restrict to the case of $\Delta = \frac{1}{2}$ and $t = 1$.

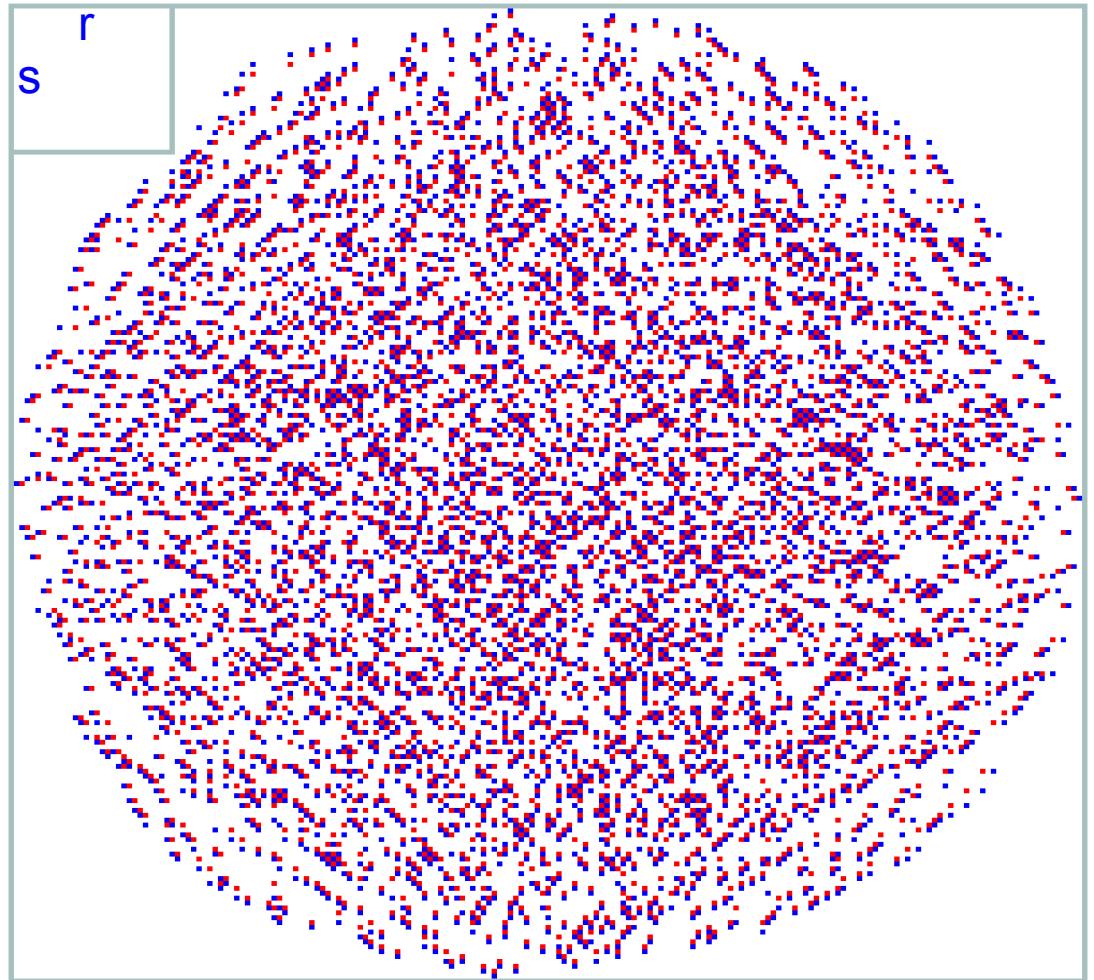
Pictures are produced with a C code based on a version kindly provided by Ben Wieland, exploiting the so-called 'Coupling From The Past' algorithm.

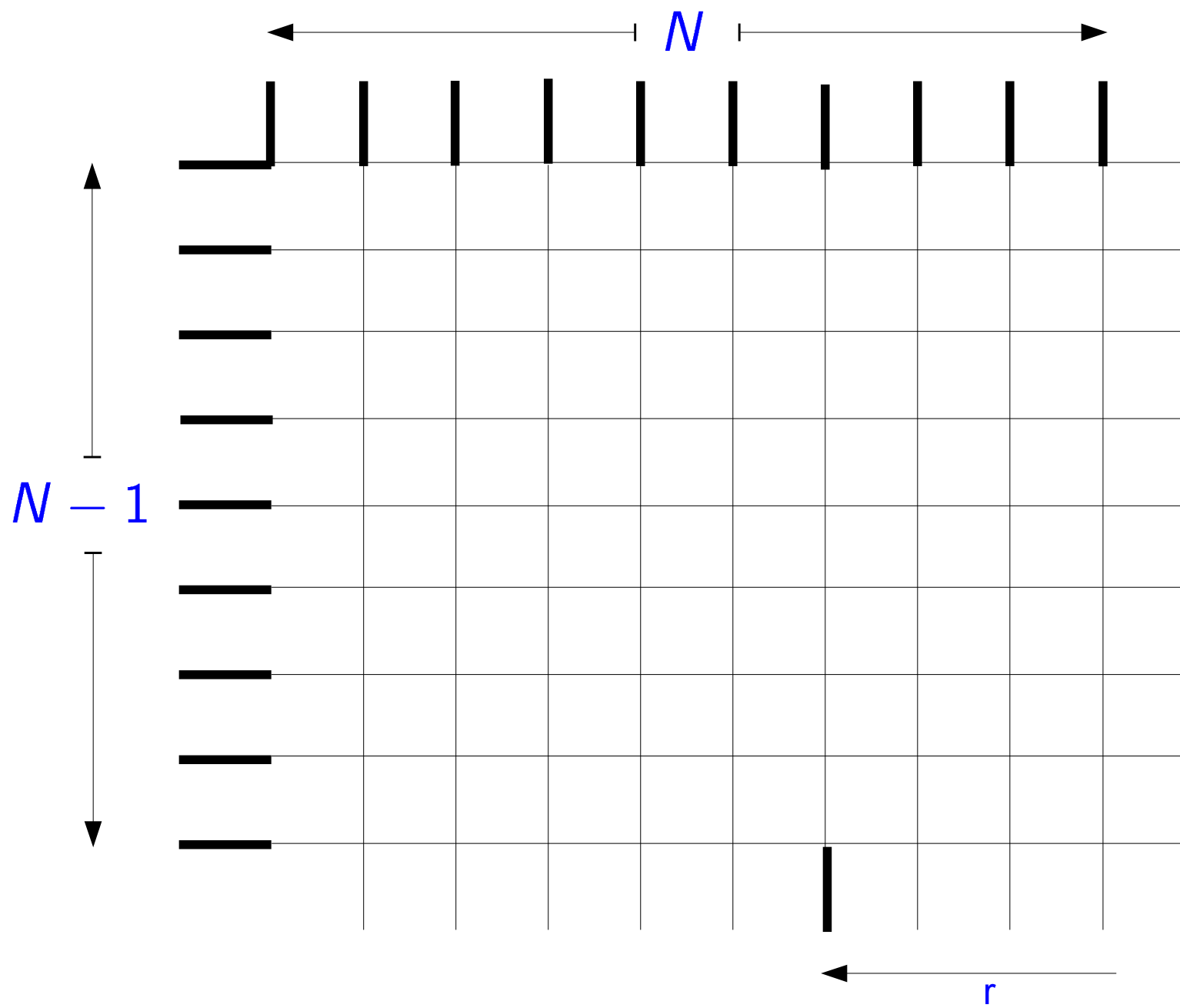
$$\Delta = \frac{1}{2}$$



 = a, b

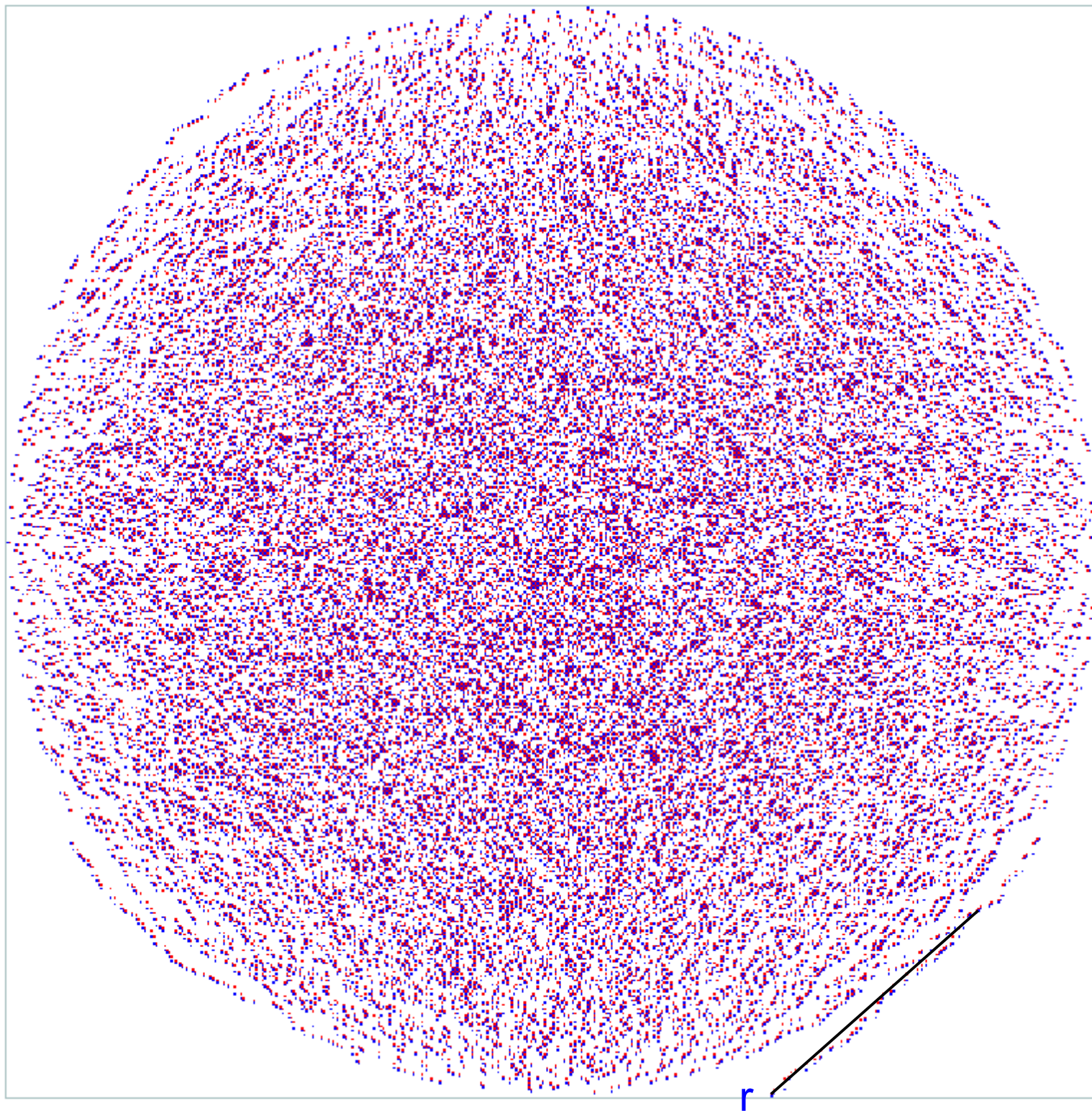
$$N = 199$$
$$r, s = 30$$





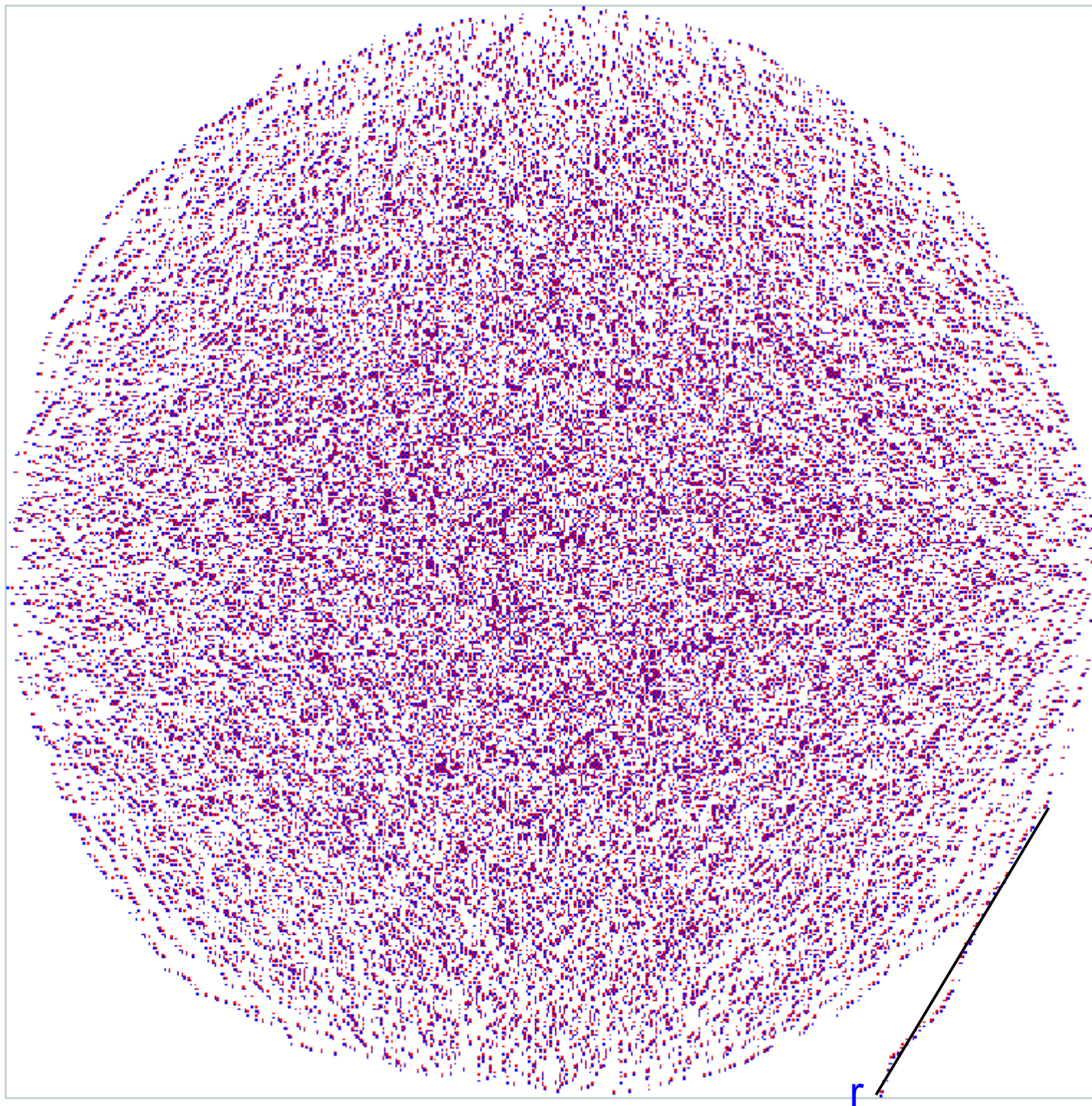
$$\Delta = \frac{1}{2}$$

$$\begin{aligned} N &= 500 \\ N' &= 499 \\ r &= 350 \end{aligned}$$



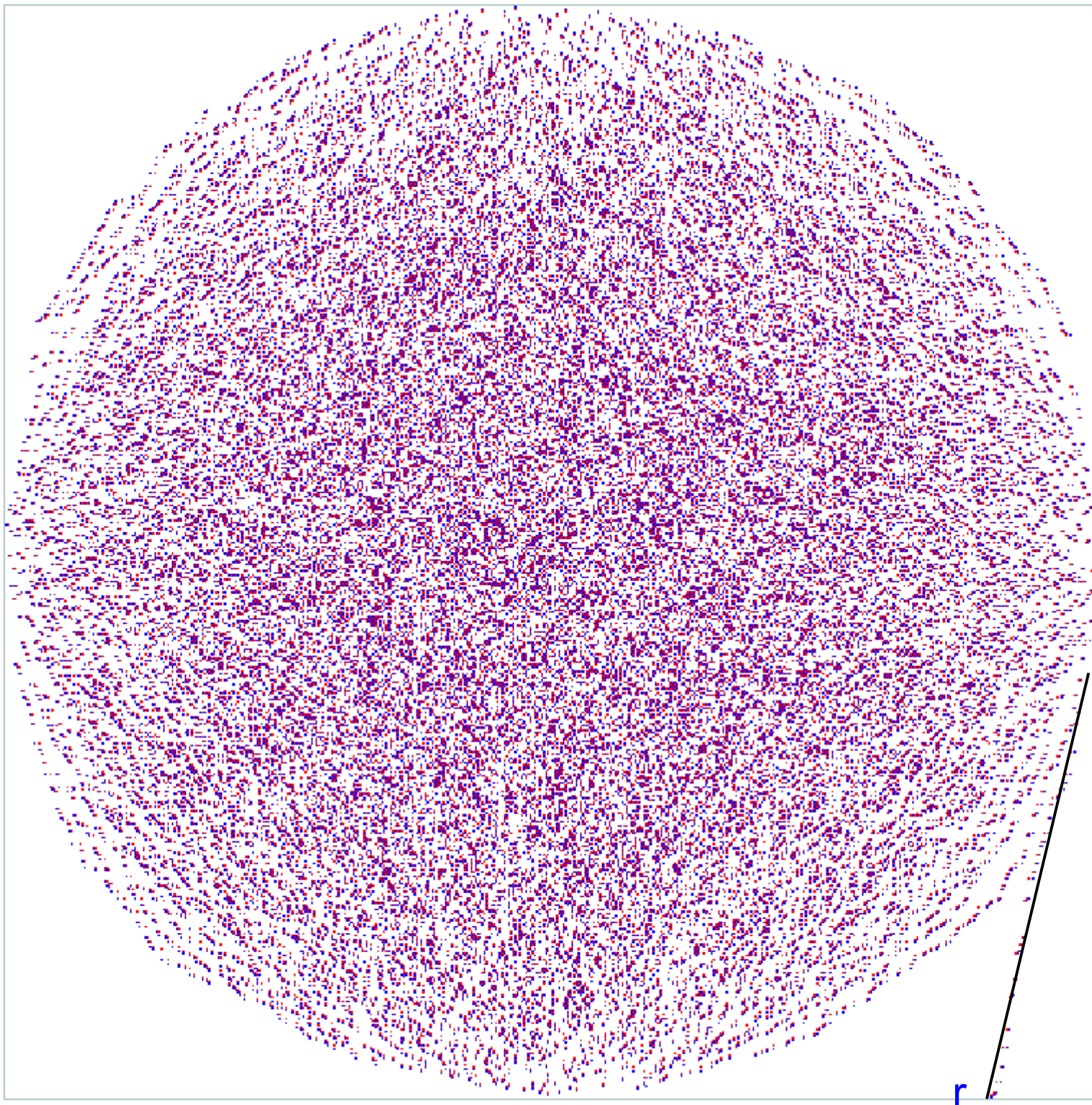
$$\Delta = \frac{1}{2}$$

$$\begin{aligned} N &= 500 \\ N' &= 499 \\ r &= 400 \end{aligned}$$



$$\Delta = \frac{1}{2}$$

$$\begin{aligned} N &= 500 \\ N' &= 499 \\ r &= 450 \end{aligned}$$

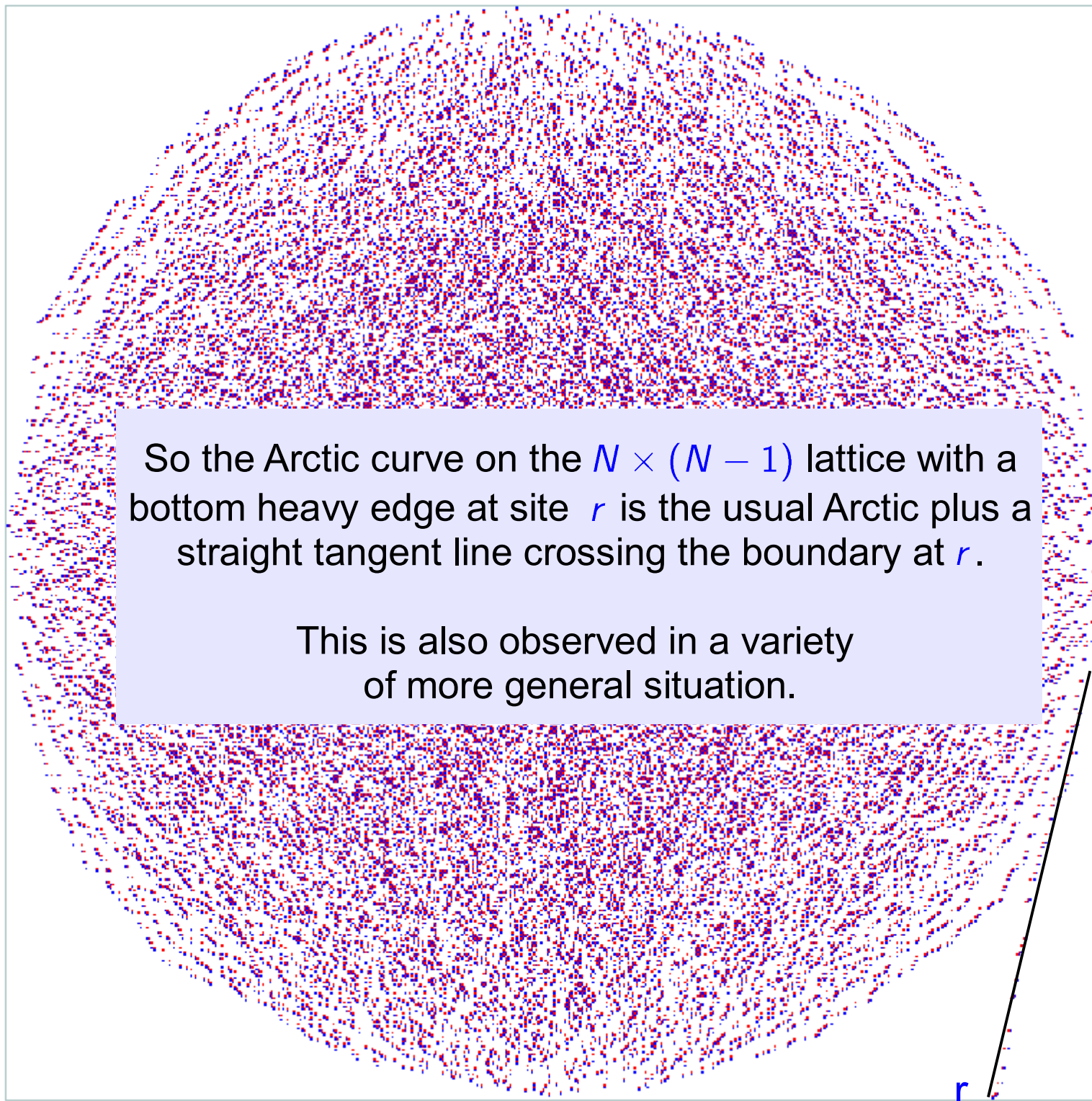


$$\Delta = \frac{1}{2}$$

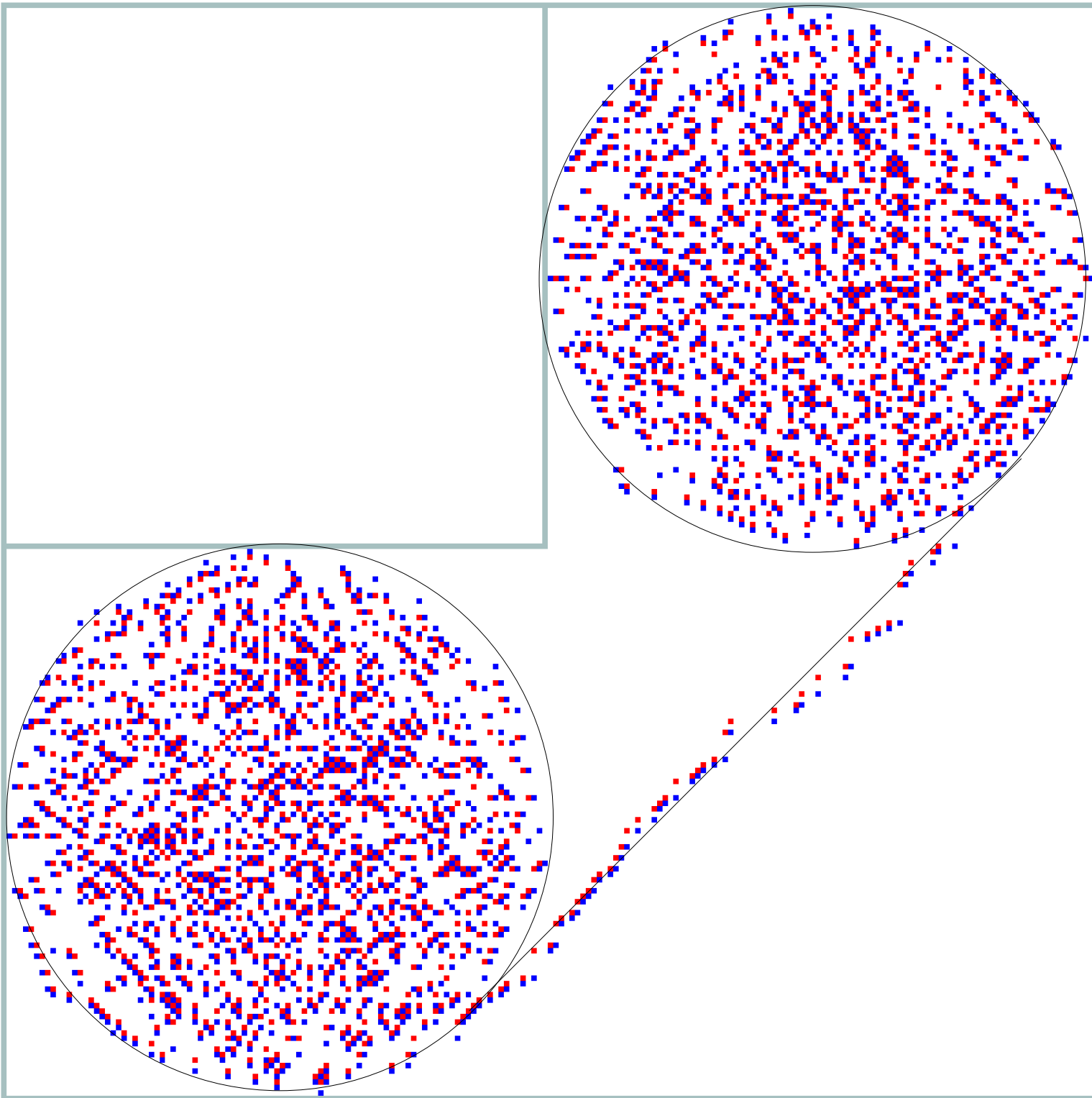
So the Arctic curve on the $N \times (N - 1)$ lattice with a bottom heavy edge at site r is the usual Arctic plus a straight tangent line crossing the boundary at r .

This is also observed in a variety of more general situation.

$$\begin{aligned} N &= 500 \\ N' &= 499 \\ r &= 450 \end{aligned}$$



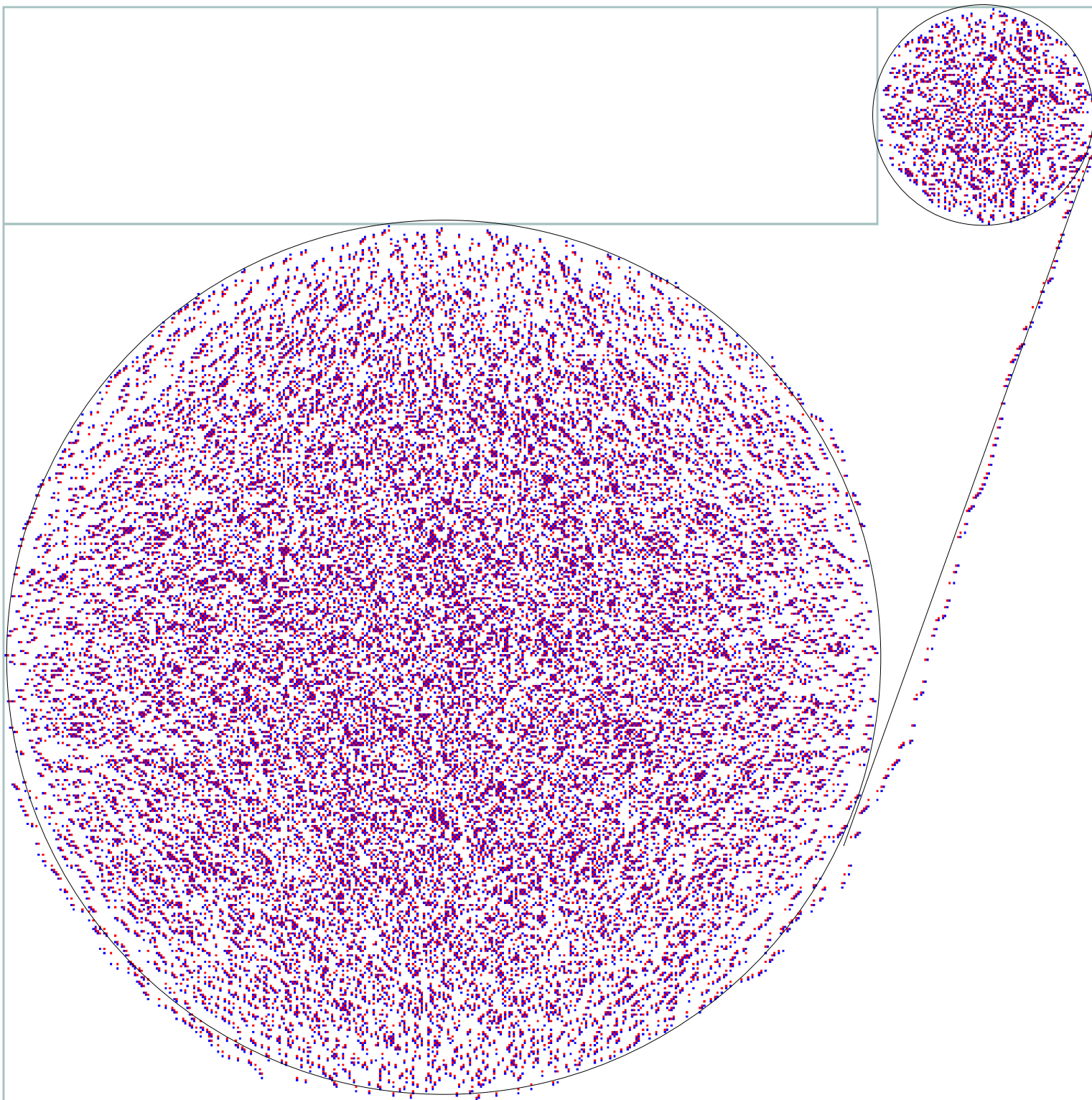
$$\Delta = \frac{1}{2}$$

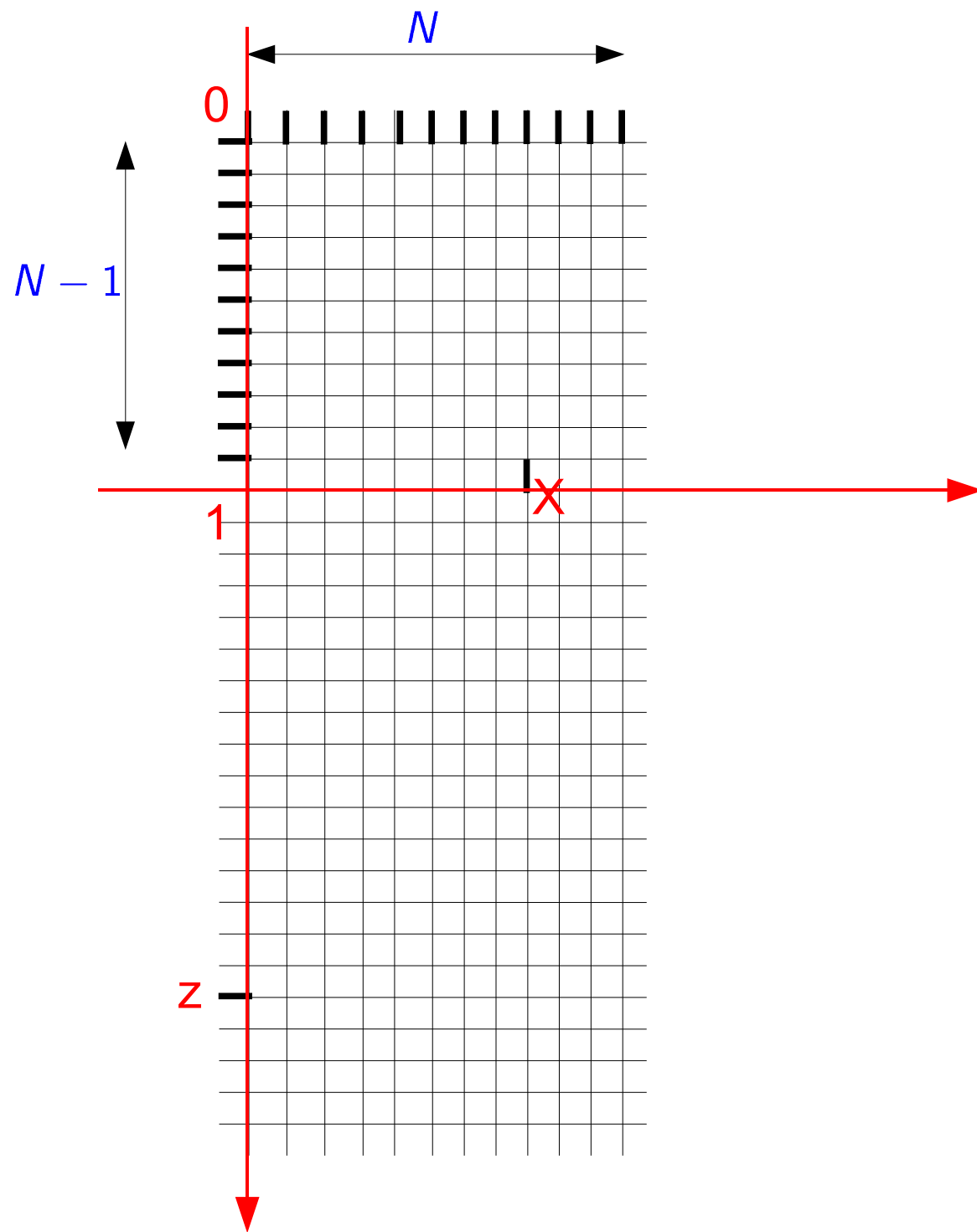


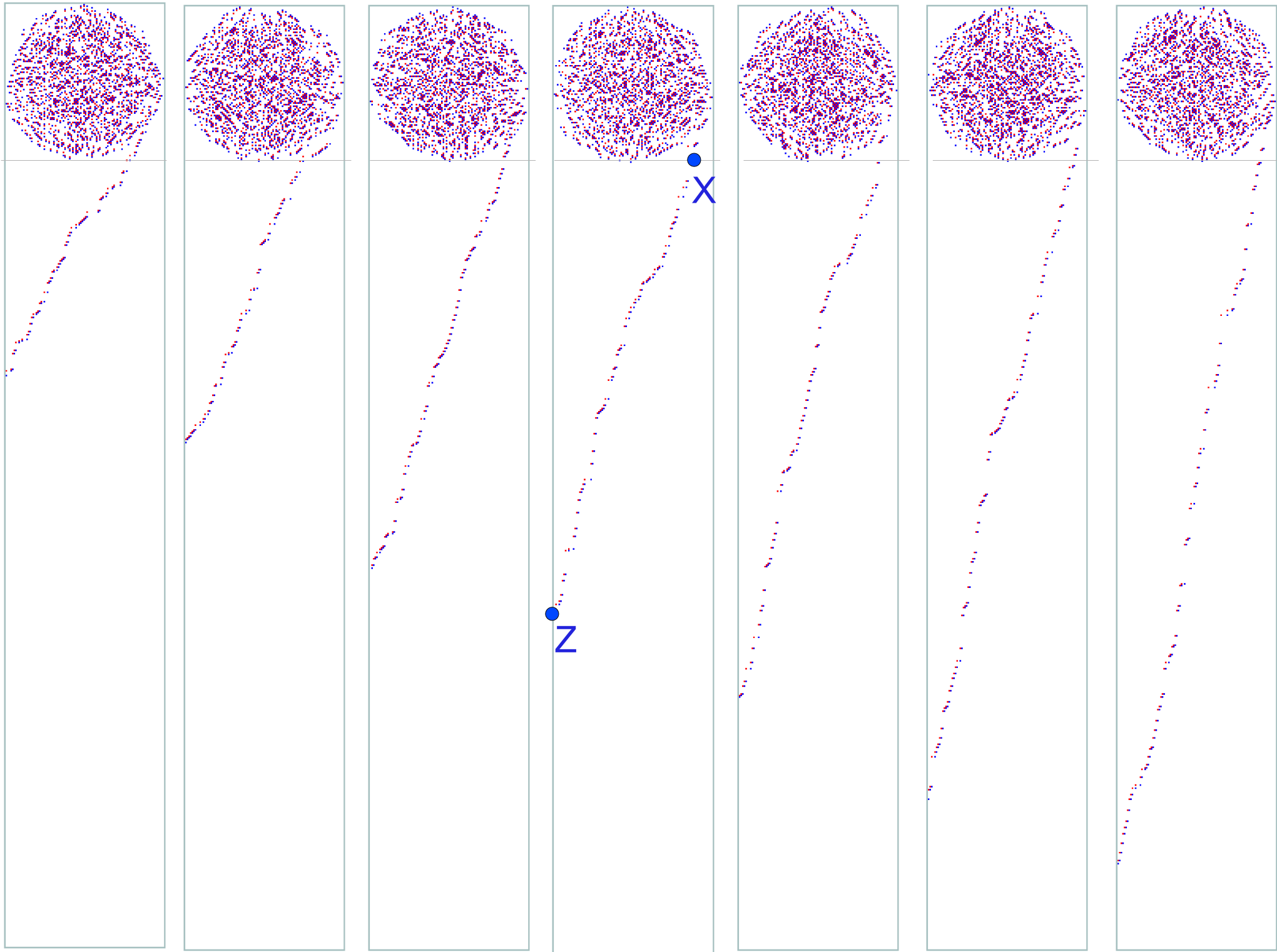
$$N = 199$$
$$r = 99$$

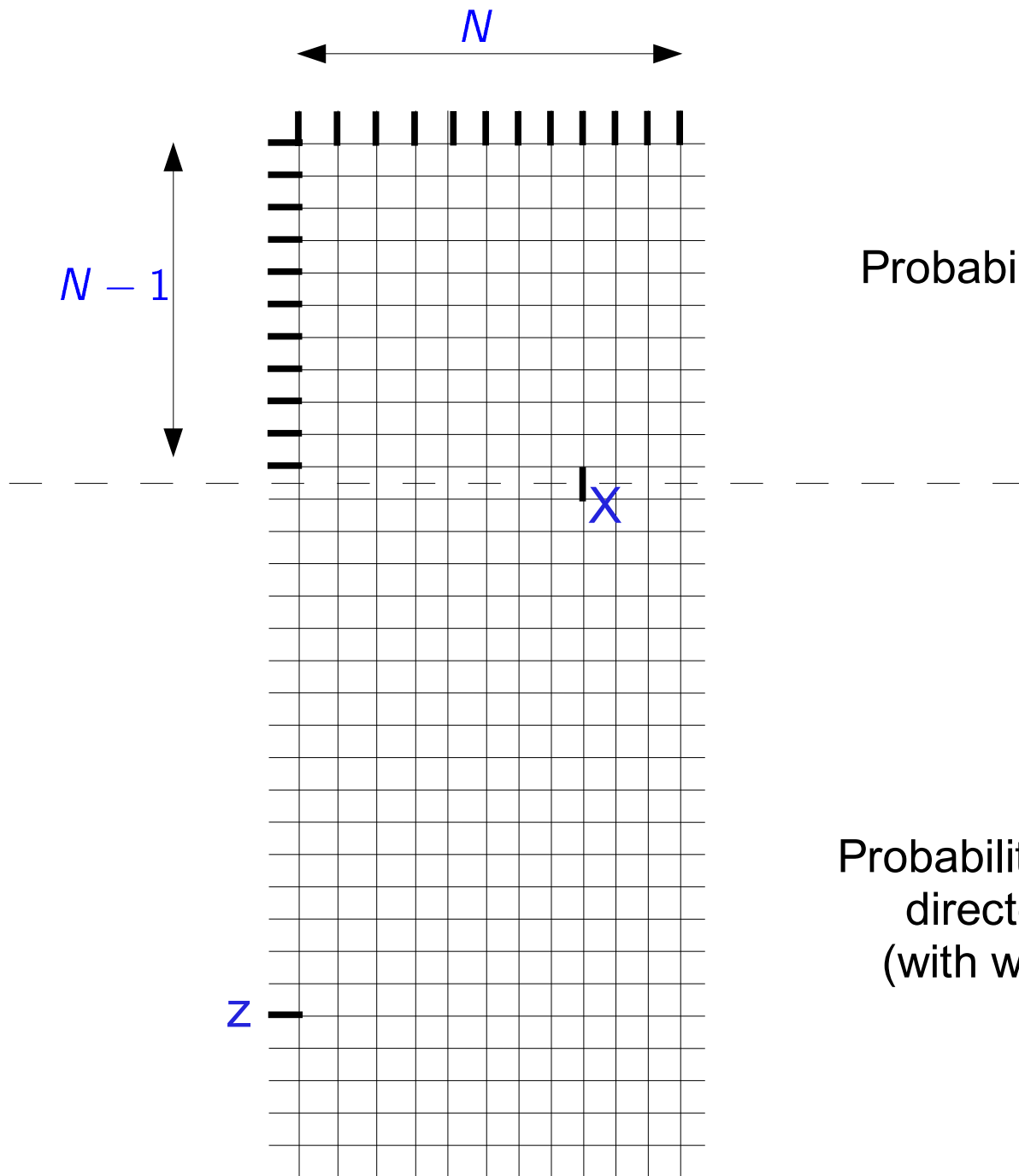
$$\Delta = \frac{1}{2}$$

$$\begin{aligned} N &= 500 \\ r &= 400 \\ s &= 99 \end{aligned}$$









Probability $\propto H_N^{(x)}$

Probability of having a weighted
directed path from X to Z
(with weights given by 6VM)

Maximizing the above probability with respect to X , one obtains a family of straight lines, parameterized by z :

$$-x \frac{1}{z} + y \frac{(t^2 - 2\Delta t + 1)}{(z - 1)(t^2 z - 2\Delta t + 1)} + \lim_{N \rightarrow \infty} \frac{1}{N} \partial_z \ln h_N(z) = 0, \quad z \in [1, +\infty)$$

which we immediately recognize! The point is that this 'geometrical' construction interpretation holds for generic domains!

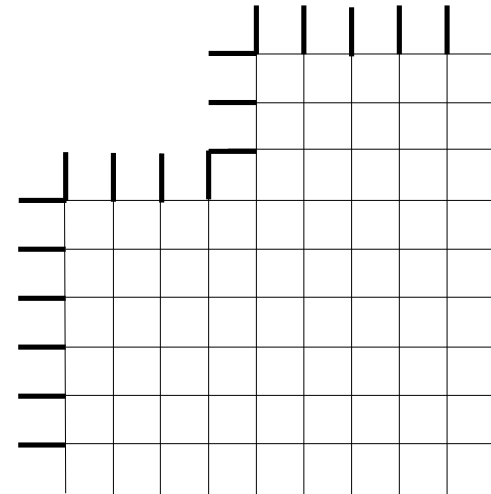
Thus on generic domains the problem of computing the Arctic curve is reduced to the evaluation of the (generating function of the) boundary correlation function, $h_N(z)$.

Does this really work?

- Checking our recipe in two cases where the boundary correlation function $h_N(z)$ is available, we have reproduced:
 - the Arctic circle of the rhombus tiling of an hexagon
 - the Arctic curve of the DW 6VM for generic values of Δ and t

- On the particular domain:

the boundary correlator $h_N(z)$ can be evaluated numerically, and the corresponding Arctic curve worked out from our recipe matches numerical data.



- For the 6VM at $\Delta = 1/2$ on a particular tetravalent graph, made of 3 bundles crossing each other (introduced in [Cantini-Sportiello'10]), the boundary correlator can be worked out thanks to a corollary of Razumov-Stroganov correspondence. The Arctic curve worked out according to the above recipe matches numerical data.

So, any further evaluation of $h_N(z)$ on different domains is welcome!

Question 1: Fluctuations

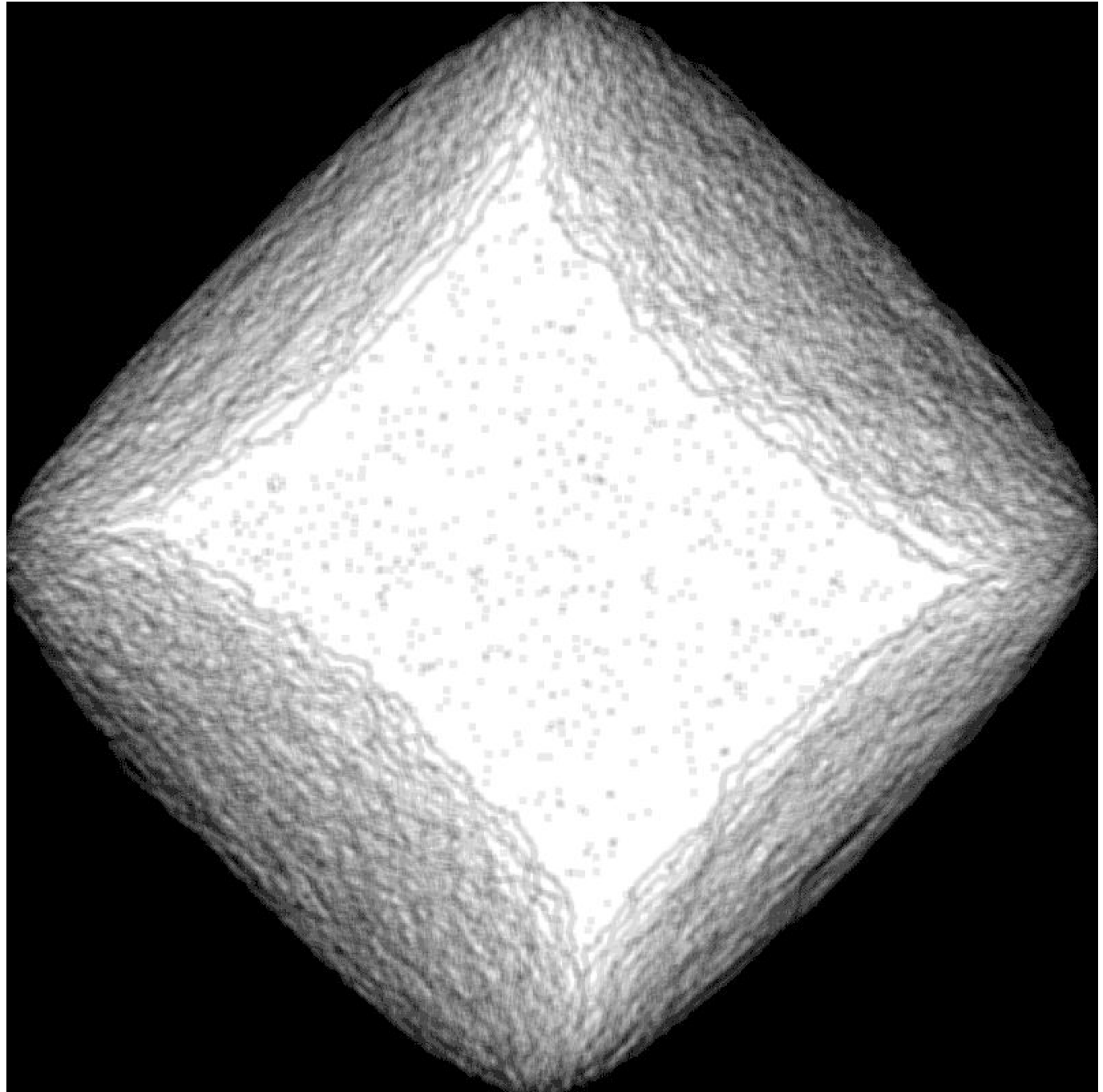
Fluctuations of the Arctic curve are driven by the evaporation of SPE solutions from the logarithmic well (Penner potential of Random Matrices), just like in the $\Delta = 0$ case. From universality considerations, Tracy-Widom behaviour, and the Airy process of the Arctic circle [Johansson'05] are again expected.

(Anti-ferroelectric regime)

$$N = 1000$$

$$\Delta = -3$$

$$t = 0.5$$



White pixels represents c -vertices

[Allison-Reshetikhin'05]

Question 1: Fluctuations

Fluctuations of the Arctic curve are driven by the evaporation of SPE solutions from the logarithmic well (Penner potential of Random Matrices), just like in the $\Delta = 0$ case. From universality considerations, Tracy-Widom behaviour, and the Airy process of the Arctic circle [Johansson'05] are again expected.

What is the D/AF phase separation curves?

What are its fluctuations?

New universality class?

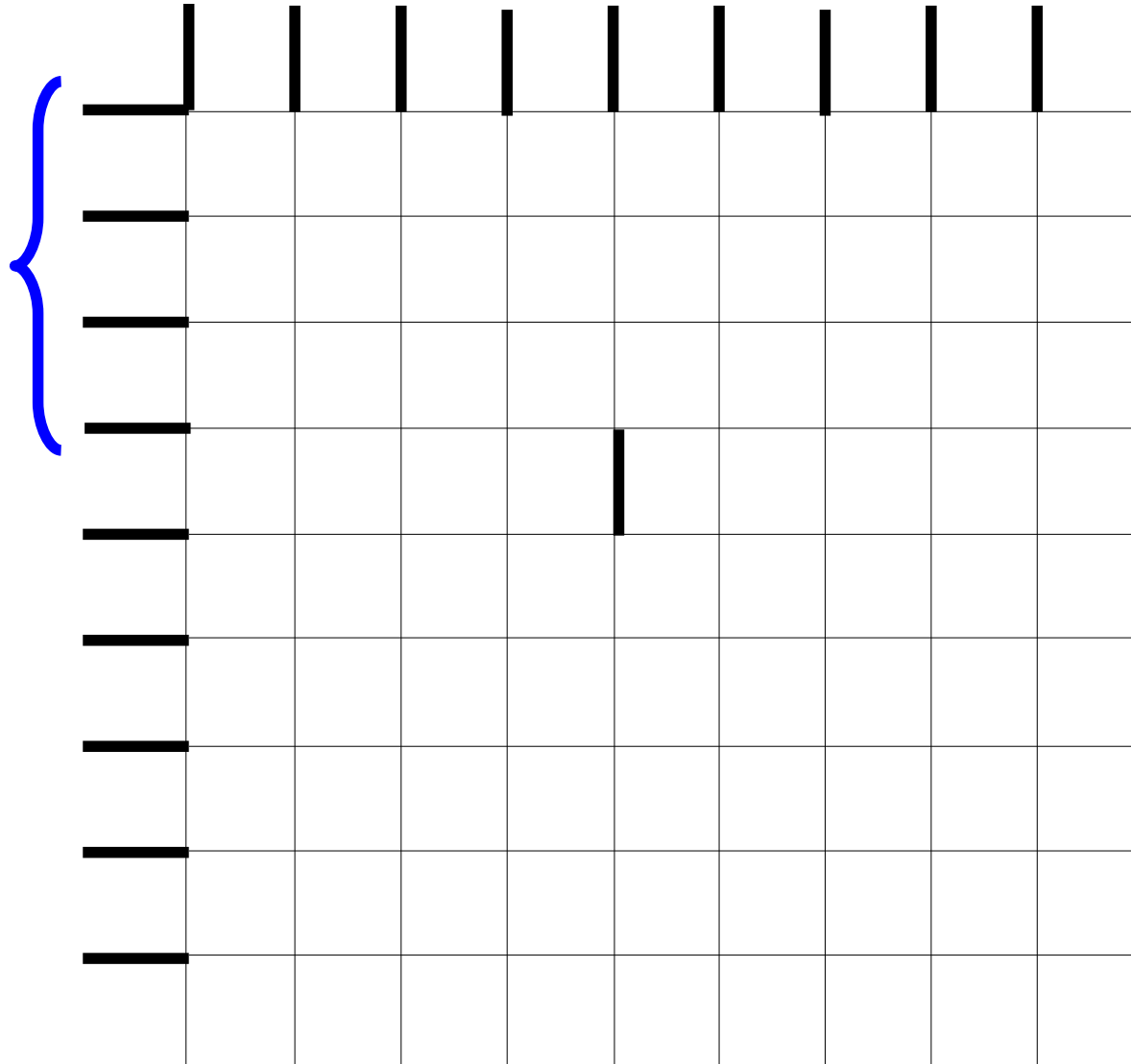
Question 2: polarization

(one-point correlation function)

$$G_N(r, s)$$

r

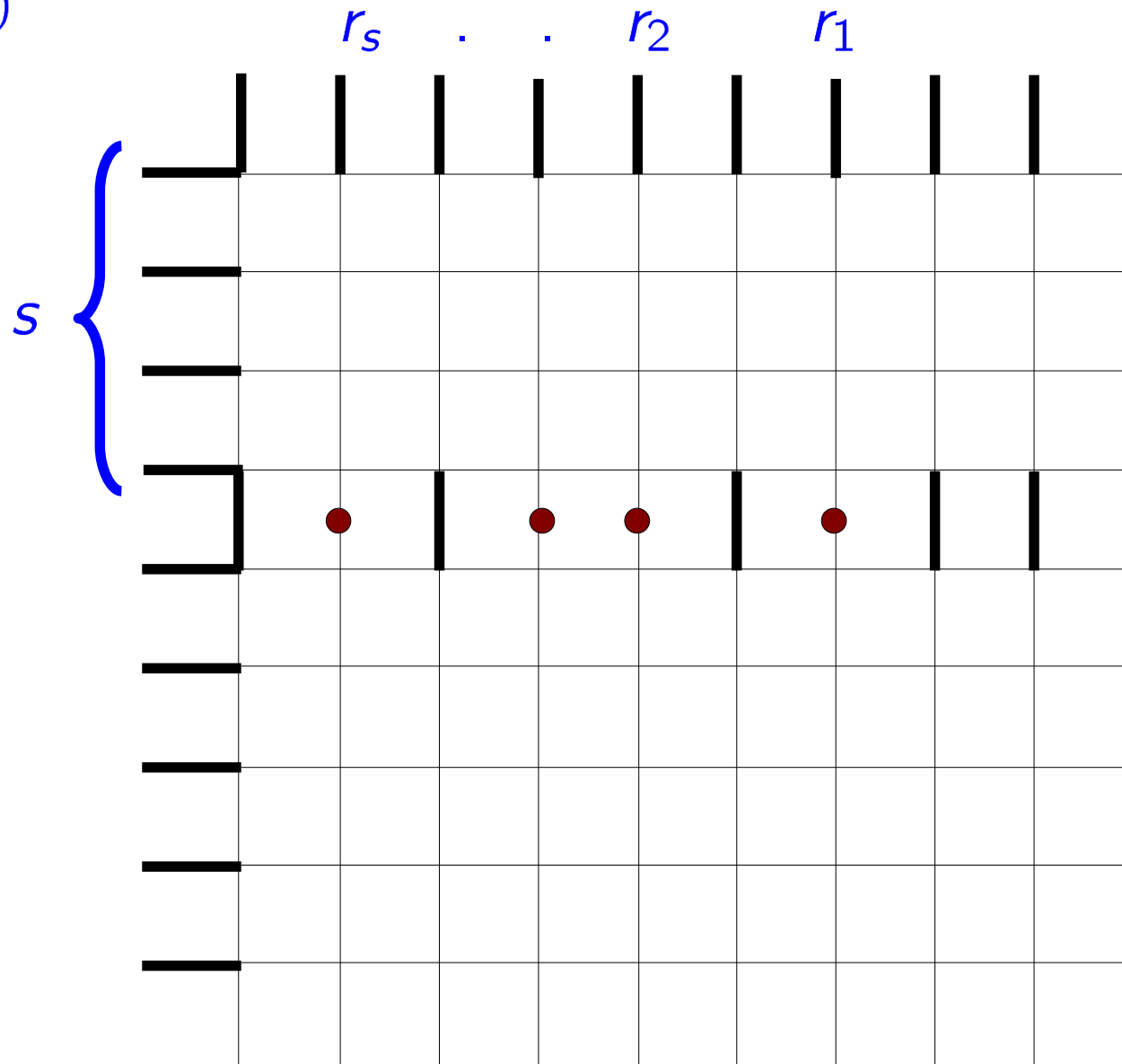
S



Row Configuration Probability

[FC-Pronko'11]

$$H_{N,s}^{(r_1, \dots, r_s)}$$

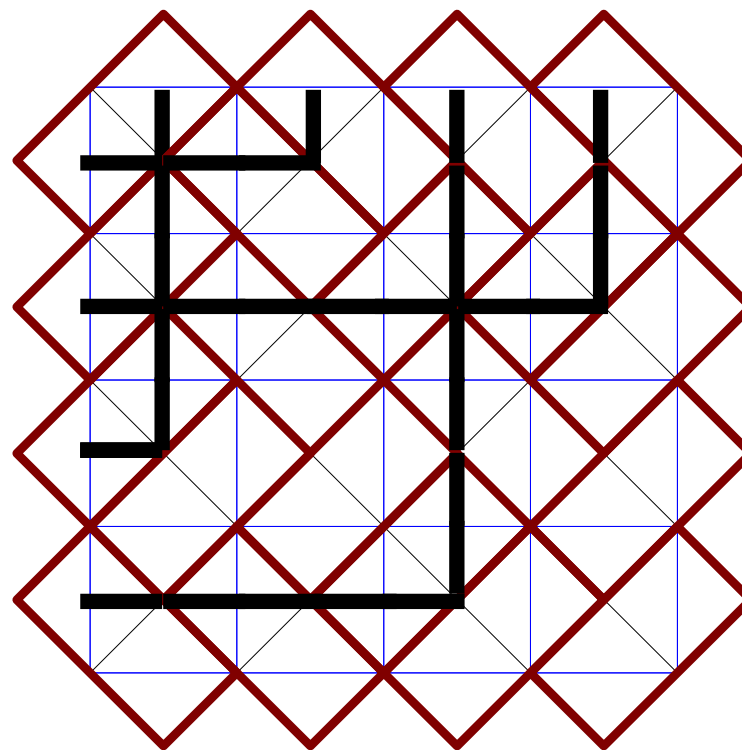
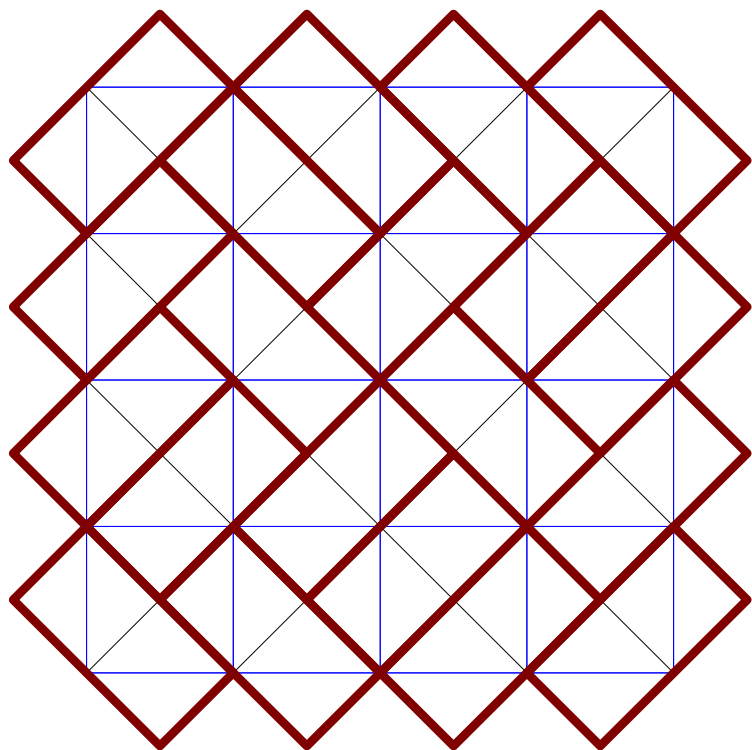


How to perform sums over $r_1, \dots, r_s$?

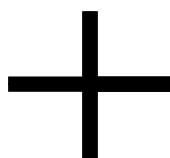
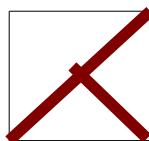
Question 3: Other approaches (CFT ?)

- At least in the disordered regime, $-1 \leq \Delta < 1$, the model is critical and has a field-theory description ($c = 1$ CFT) in the scaling limit.
- Of course, the boundary condition is not conformal at all; nevertheless a field-theory approach could provide a useful, and perhaps even more powerful alternative.

How to rephrase our problem in field-theory language?

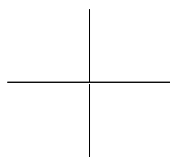
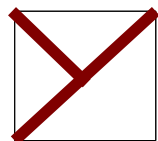


a



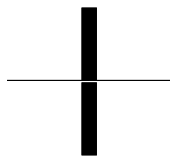
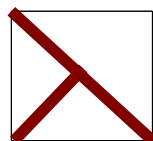
a

a



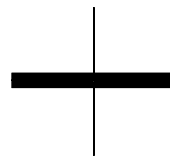
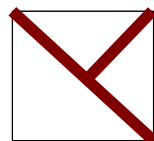
a

b



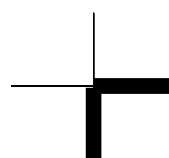
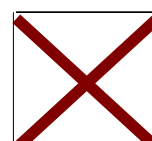
b

b

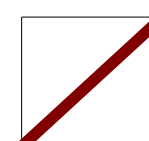
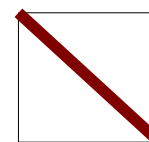


b

1



c



} $c^2/2$

c