Analytic Results for Quantum Quenches in the Transverse Field Ising Chain

Fabian Essler (Oxford)

Pasquale Calabrese (Pisa)
Maurizio Fagotti (Pisa)

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Quantum systems out of equilibrium

Idea:

A. Consider a quantum many-particle system with Hamiltonian H

B. Prepare the system in a state $|\psi\rangle$ that is **not** an eigenstate.

C. Time evolution $|\psi(t)\rangle = \exp(-iHt)\,|\psi\rangle$

D. Study time evolution of local observables $\langle\psi(t)|O(x)|\psi(t)\rangle$ in the **thermodynamic limit**.

Quantum systems out of equilibrium

Why is this interesting?

A. It explores a new regime of Quantum Theory: properties of states other than ground state/equilibrium states + their immediate vicinities (low lying excitations).

B. In the **thermodynamic limit** new physics may occur.

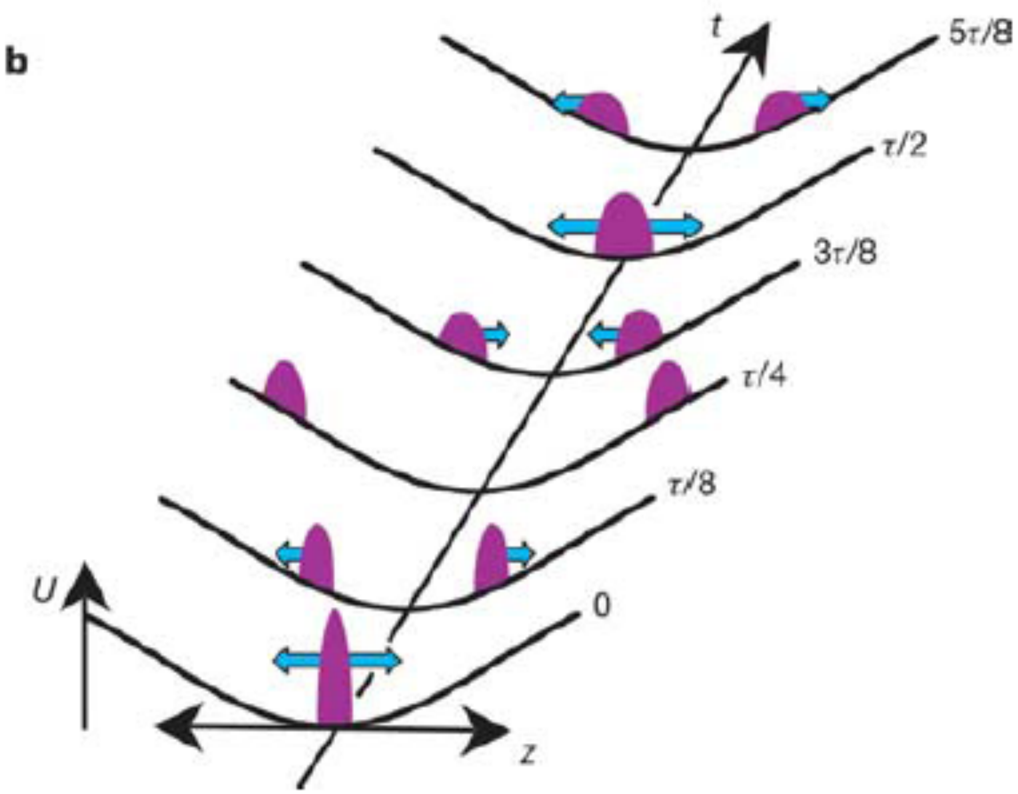
C. Is the time evolution towards a **stationary state**?

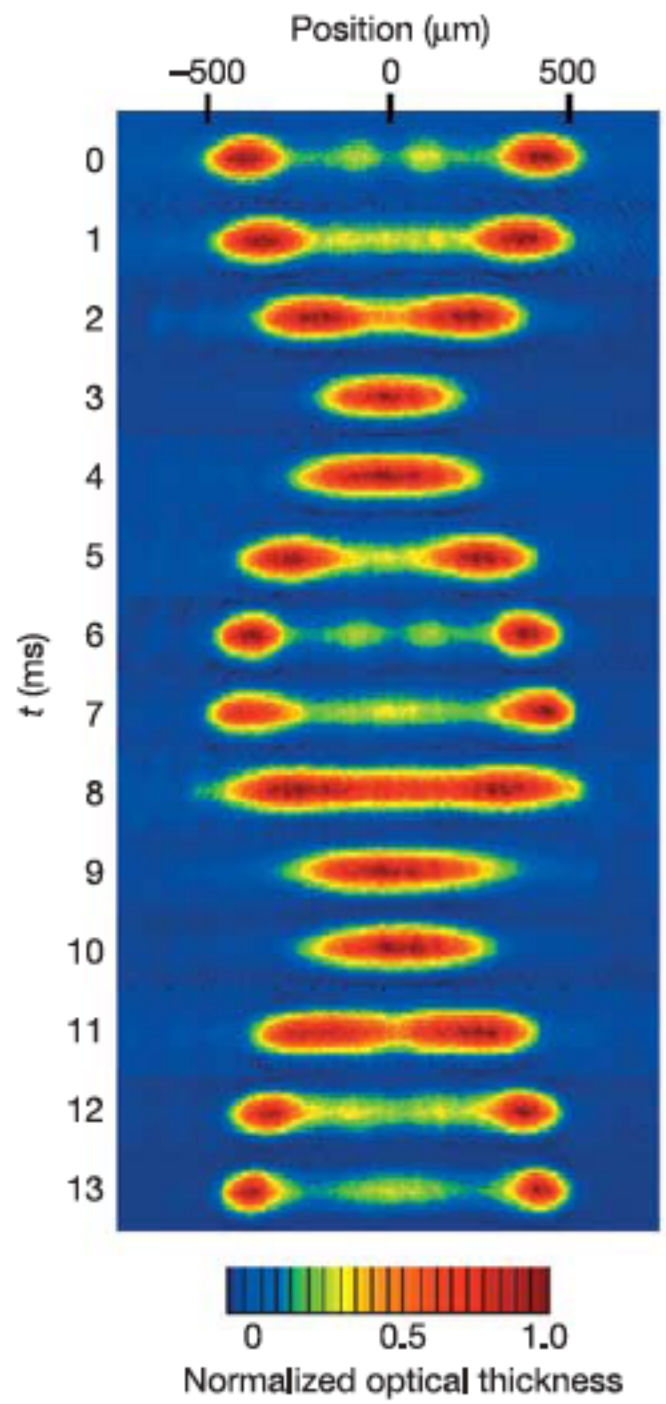
D. If so, is there a general principle that tells us how to evaluate averages of observables in these states (c.f. Gibbs distribution).

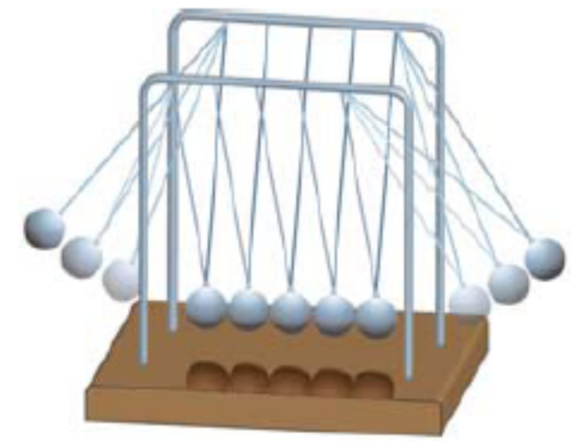
Experiments: "Quantum Newton's Cradle"

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

40-250 ^{87}Rb atoms in a 1D optical trap

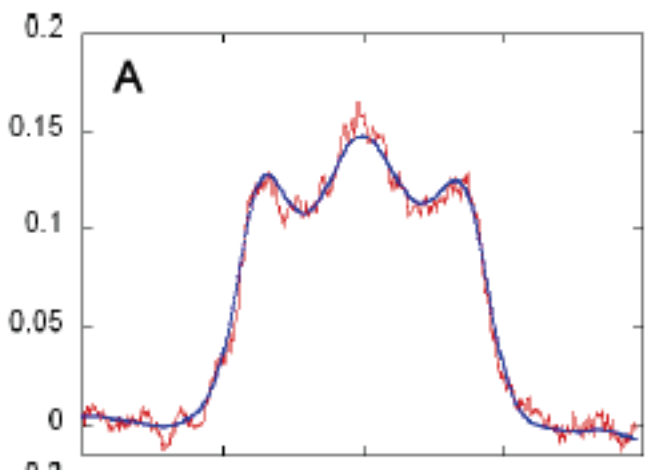






Essentially unitary time evolution.

- 1D system "relaxes" very slowly in time, to a strange distribution.

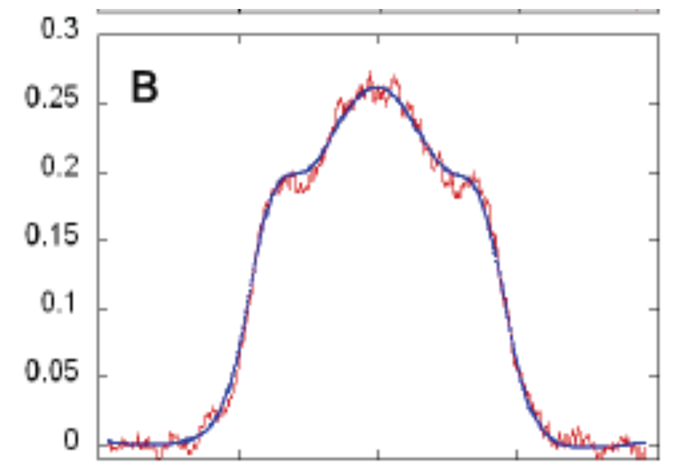


Position (μm)

γ=18

τ_{th}

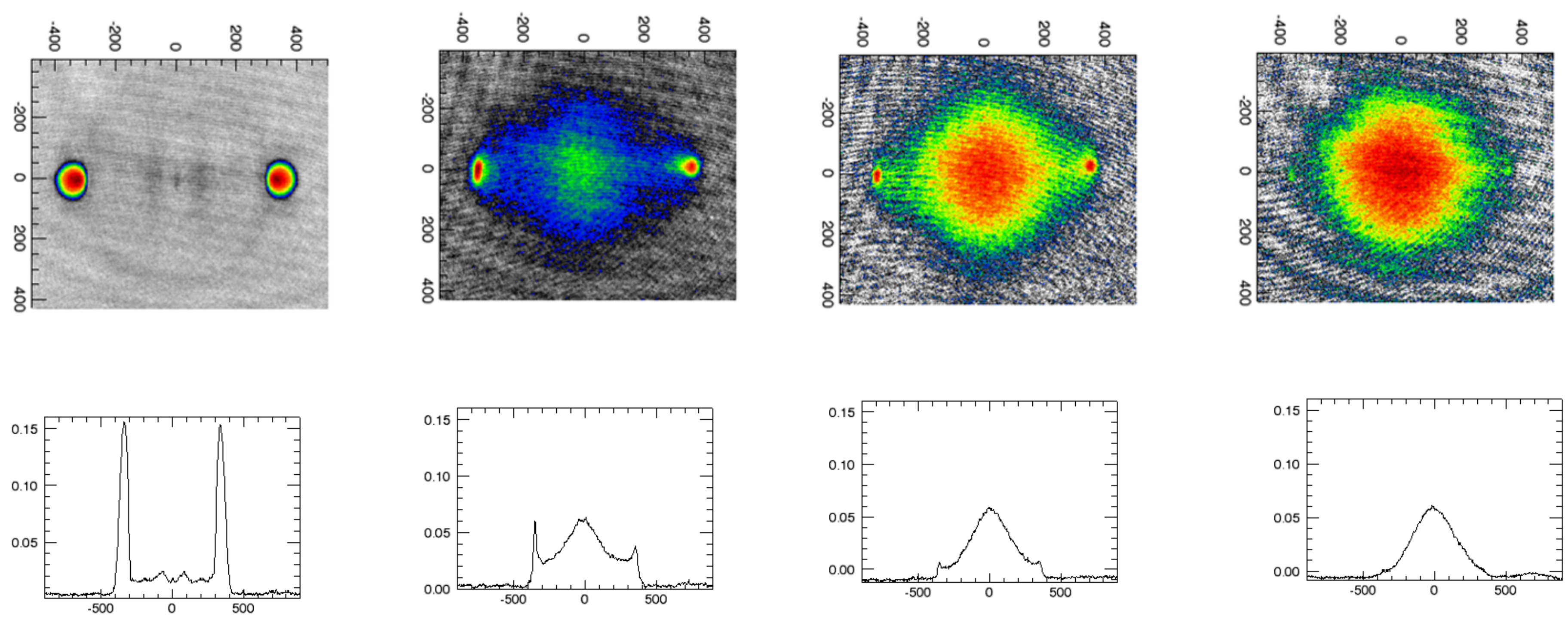
>390τ



γ=3.2

>1910τ

- 2D and **3D** systems relax quickly: Thermalization occurs in ~3 collisions.



0τ 2τ 4τ 9τ

"Quantum Newton's Cradle"

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

Suggestion: the 1D case is special because the system is "close to being integrable"

Without trap:

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{N \geq j > k \geq 1} \delta(x_j - x_k).$$

Has infinite number of local higher conservation laws, solvable by Bethe Ansatz (Lieb+Liniger '63)

My opinion: experiments not sensitive to integrability, but to **dimensionality** (difficulty of mtm relaxation in D=1)

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Question: is the nonequilibrium evolution of integrable models special and if so, how?

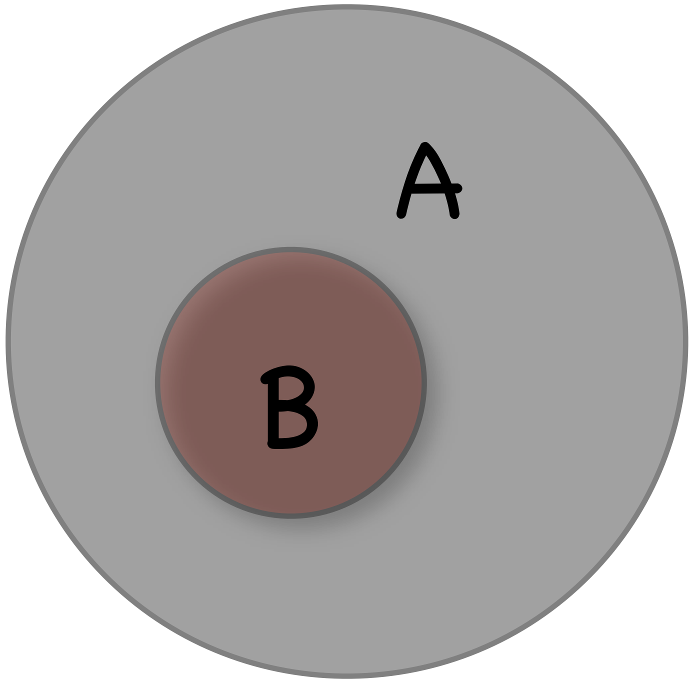
"Thermalization"

(Deutsch '91, Srednicki '94)

Belief: "generic" system "thermalize" at infinite times.

Density matrix: $\rho=|\psi\rangle\langle\psi|$

Reduced density matrix: $\rho_B=\mathrm{tr}_A\,|\psi\rangle\langle\psi|$



If A is infinite then $\rho_B=\exp(-\beta_{eff}\,H_B)/Z_B$

Rigol et al (2007): **Integrable systems do not thermalize.**

Let I_m be local integrals of motion $[I_m, I_n]=0$

Late time behaviour described by **generalized Gibbs ensemble:**

$$\rho_{gG}=\exp(-\Sigma\ \lambda_m\ I_m)/Z_{gG}$$

$$Z_{gG}=\mathrm{tr}\ \exp(-\Sigma\ \lambda_m\ I_m)$$

$$\lim_{t\to\infty}\ \langle\psi(t)|O(x)|\psi(t)\rangle\ =\mathrm{tr}[\rho_{gG}\ O(x)]$$

λ_m fixed by $\quad \mathrm{tr}[\rho_{gG}\ I_m]=\ \langle\psi(0)|\ I_m\ |\psi(0)\rangle$

Transverse Field Ising Chain

Simplest paradigm of a T=0 **Quantum Phase Transition**

Hamiltonian:

$$H = -J \sum_{j=1}^{L} \left[\sigma_j^z \sigma_{j+1}^z + h \sigma_j^x \right].$$

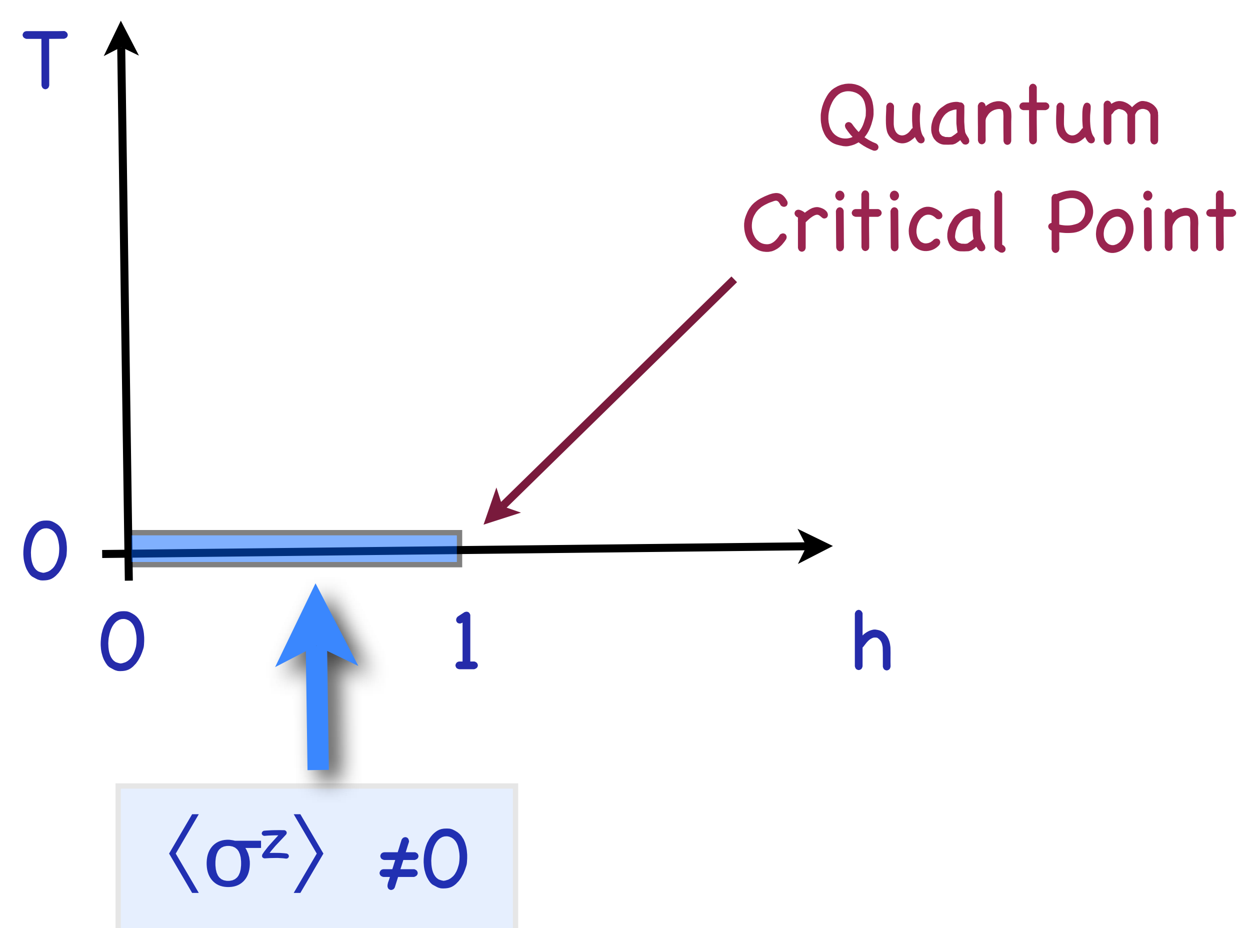
$\mathbb{Z}_2$ symmetry: rotation by π around x-axis.

$$\sigma_j^\alpha \rightarrow -\sigma_j^\alpha \,, \quad \alpha = y, z.$$

Phase Diagram:

order parameter: $\langle \sigma_j^z \rangle$

($\langle \sigma_j^x \rangle \neq 0$ always)



Transverse Field Ising Chain

Jordan-Wigner transformation to spinless fermions:

$$\sigma^x_j = 1 - 2c^\dagger_j c_j \ ,$$

local

$$\sigma^z_j = -\prod_{l<j}(1 - 2c^\dagger_l c_l)(c_j + c^\dagger_j).$$

nonlocal

Fourier+Bogoliubov transformations:

$$c(k) = \frac{1}{\sqrt{L}}\sum_j c_j \ e^{-ikj}.$$

$$\begin{pmatrix} c(k) \\ c^\dagger(-k) \end{pmatrix} = R_h(k)\begin{pmatrix} \alpha_k \\ \alpha^\dagger_{-k} \end{pmatrix}$$

$$H = \sum_k \epsilon_h(k)\left[\alpha^\dagger_k \alpha_k - \frac{1}{2}\right]$$

$$\epsilon_h(k) = 2J\sqrt{1 + h^2 - 2h\cos(k)}.$$

Ground State:

$$\alpha_k|0\rangle = 0.$$

This will be our initial state

Quantum Quench h→h'

New Hamiltonian: $$H(h') = \sum_k \epsilon_{h'}(k)\left[\beta_k^\dagger \beta_k - \frac{1}{2}\right]$$

Time evolution: $$\beta_k(t) = e^{-i\epsilon_{h'}(k)t}\beta_k$$

New vs old Bogoliubov fermions: $$\begin{pmatrix}\beta_k \\ \beta^\dagger_{-k}\end{pmatrix} = U(k)\begin{pmatrix}\alpha_k \\ \alpha^\dagger_{-k}\end{pmatrix} \qquad U(k) = R^\dagger_{h'}(k)R_h(k)$$

Time evolution of σ^x

(Barouch, McCoy & Dresden '70)

$$\sigma^x_j(t) = -\frac{1}{L}\sum_{k,p} e^{-i(k-p)j}\ (\alpha^\dagger_k, \alpha_{-k})\ S(k,p,t)\begin{pmatrix}\alpha_p \\ \alpha^\dagger_{-p}\end{pmatrix}$$

S(k,p,t) a known 2x2 matrix.

$L \to \infty$ ↓ straightforward calculation

$$\langle 0|\sigma_j^x(t)|0\rangle \xrightarrow[t\to\infty]{} C(h') + O(t^{-3/2}).$$

Thermalization?

Gibbs ensemble with $T_{\rm eff}$ $$\rho_{\rm G} = \frac{1}{\mathcal{Z}} e^{-H(h')/T_{\rm eff}}$$

$T_{\rm eff}$ fixed by $$\mathrm{tr}\left[\rho_{\rm G}\ H(h')\right] = \langle 0|H(h')|0\rangle$$

$$\lim_{t\to\infty} \mathrm{tr}\left[\rho_G\ \sigma^x(t)\right] \neq C(h').$$

→ **no thermalization.**

How about the Generalized Gibbs Ensemble?

Conserved Quantities: $I(k) = \beta_k^\dagger \beta_k$

Density Matrix: $\rho_{gG} = \dfrac{1}{\mathcal{Z}_{gG}} e^{-\sum_k \lambda_k I(k)}$

Fix Lagrange Multipliers: $\langle 0 | I(k) | 0 \rangle = \mathrm{tr}\left[\rho_{\mathrm{gG}}\ I(k)\right]$

Like a **mode-dependent temperature** $\rho_{\mathrm{gG}} = \frac{1}{\mathcal{Z}_{\mathrm{gG}}} e^{-\sum_k H(k)/T_{\mathrm{eff}}(k)}$

$$\lim_{t\to\infty} \mathrm{tr}\left[\rho_{gG}\ \sigma^x(t)\right] = C(h').$$

$\Rightarrow$ **GGE works.**

The role of locality

(Rossini et al'10)
(Fioretto and Mussardo '10)

σ^x is quite **special** (non generic): it is **local** w.r.t. to the fermion excitations and couples only to 2-particle states.

σ^z is **non-local** (couples to states with arbitrary number of fermions) and it's difficult to say from numerical studies whether 2-point function **thermalizes** (particularly for small quenches).

Is it possible that certain operators integrable models thermalize and others don't?

Our work: 1 and 2-point functions of σ^z

Calculations are **difficult**. Developed two analytic methods based on (a) determinants (b) form factors.

Result 1: t=∞ behaviour for arbitrary h,h'

$$\lim_{t\to\infty} \langle 0|\sigma_j^z(t)\ \sigma_{j+\ell}^z(t)|0\rangle \sim \exp\left(-\ell/\xi\right)\ ,\ell \gg 1,$$

ξ a simple function of h,h':

$$\xi^{-1} = \begin{cases} -\ln\left[x_+ + x_- + \sqrt{4x_+x_-}\right] & \text{if } h,h' < 1 \\ \ln\left(\min[h,h_1]\right) - \ln\left[x_+ + x_- + \sqrt{4x_+x_-}\right] & \text{if } h,h' > 1 \\ -\ln\left[x_+ + x_-\right], & \text{else.} \end{cases}$$

$$x_\pm = \frac{1}{4}[\min(h',h'^{-1}) \pm 1][\min(h,h^{-1}) \pm 1] \qquad h_1 = \frac{1 + h'h + \sqrt{(h'^2-1)(h^2-1)}}{h'+h}$$

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Compatible with **GGE**, but **not with thermalization!**

Interpretation in terms mode-dependent temperature: (c.f. Calabrese/ Cardy '07)

$$\xi^{-1} = \int_0^\pi \frac{dk}{\pi}\xi^{-1}(k) = \int_0^\pi \frac{dk}{\pi} \ln\left[\tanh\frac{\epsilon_{h'}(k)}{2T_{\rm eff}(k)}\right] . \qquad (h,h'<1)$$

thermal correlation length:

$$\xi_T^{-1} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln\left|\tanh\frac{\epsilon_{h'}(k)}{2T}\right| .$$

Result 2: Time dependence for late times

Quenches within the ordered phase (h<1 to h'<1):

$$\langle 0|\sigma_j^z(t)|0\rangle \sim \exp\left(t\int_0^\pi \frac{dk}{\pi}\epsilon_h'(k)\ln\left[\cos(\Delta_k)\right]\right)$$

$$\cos\Delta_k = \frac{h'h-(h'+h)\cos k+1}{\epsilon_h(k)\epsilon_{h'}(k)}.$$

(approaches **zero** although we remain in the ordered phase).

Mode-dependent decay rate: $\tau^{-1} = \int_0^\pi \frac{dk}{\pi}\tau^{-1}(k) = \int_0^\pi \frac{dk}{\pi}\epsilon_{h'}(k)\ln\left[\cos\Delta_k\right].$

Decay rate and correlation length related by

$$\xi(k) = \epsilon_{h'}(k)\,\tau(k).$$

Can understand approach to stationary state in terms of GGE

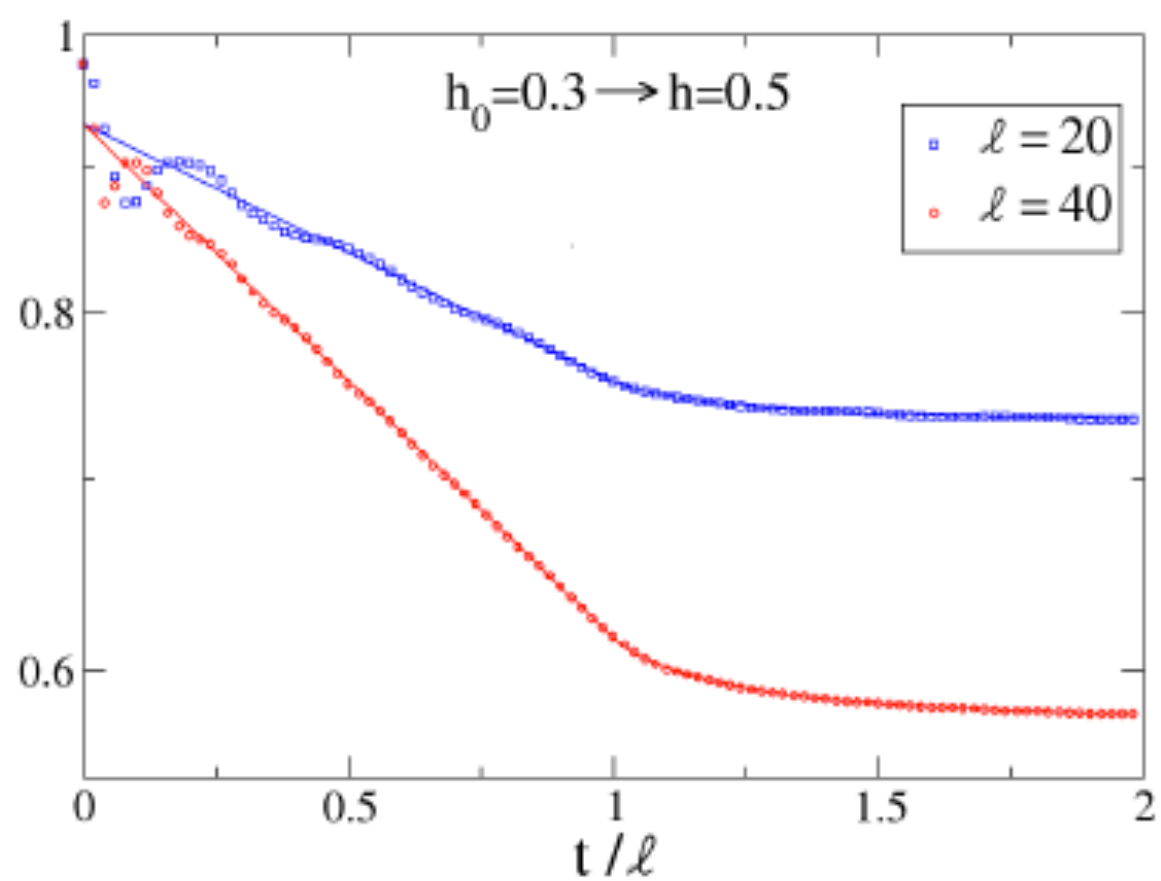
Quenches within the ordered phase (h<1 to h'<1):

$$\langle 0|\sigma_j^z(t)\ \sigma_{j+\ell}^z(t)|0\rangle \sim \exp\left(t \int_{2t\epsilon'_{h'}(k)<\ell} \frac{dk}{\pi} 2\epsilon'_{h'}(k) \ln\left[\cos(\Delta_k)\right] + \ell \int_{2t\epsilon'_{h'}(k)>\ell} \frac{dk}{\pi} \ln\left[\cos(\Delta_k)\right] \right)$$

Asymptotics

vs

Numerics:



Approach I: Block-Toeplitz Determinants

Express $\sigma_j^z(t)$ in terms of the "old" Bogoliubov fermions α_k

$$\langle 0|\sigma_j^z(t)\ \sigma_{j+n}^z(t)|0\rangle \xrightarrow{\text{Wick's thm}} \mathrm{Pf}(T)$$

$$T_{ln} = \begin{pmatrix} f_{l-n} & -g_{n-l} \\ g_{l-n} & -f_{l-n} \end{pmatrix}$$ Block-Toeplitz matrix

$$f_l \ = \ i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikl} \sin\left(\Delta_k\right)\ \sin\left(2\epsilon_h'(k)t\right)$$

$$g_l \ = \ \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ik(l-1)} \left[\cos\left(\Delta_k\right) + i\sin\left(\Delta_k\right)\cos\left(2\epsilon_h'(k)t\right)\right]$$

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$\downarrow$ Multi-dim stationary phase approx.

$$\exp\left(t \int_{2t\epsilon'_{h'}(k)<n} \frac{dk}{\pi} 2\epsilon'_{h'}(k) \ln\left[\cos(\Delta_k)\right] + n \int_{2t\epsilon'_{h'}(k)>n} \frac{dk}{\pi} \ln\left[\cos(\Delta_k)\right] \right)$$

Approach II: "Form-Factor" Sums

Consider a quench within the ordered phase h,h'<1

1. Go to large, finite volume L

2. initial state = one of the two ground states

$$|0\rangle = \frac{1}{\sqrt{2}}\Big[|0\rangle_{\rm R} \pm |0\rangle_{\rm NS}\Big] \qquad \alpha_q|0\rangle_{\rm R} = 0 \qquad q_m = \frac{2\pi}{L}m\,, \quad m = -\frac{L}{2},\ldots,\frac{L}{2}-1$$

periodic bc's on fermions

antiperiodic bc's on fermions

3. Express this in terms of the new Bogoliubov fermions

$$|0\rangle_{\rm NS} = \exp\left(i\sum_{p>0} K(q)\beta^\dagger_q\beta^\dagger_{-q}\right)|0'\rangle_{\rm NS}\,,$$

$$|0\rangle_{\rm R} = \exp\left(i\sum_{q>0} K(q)\beta^\dagger_q\beta^\dagger_{-q}\right)|0'\rangle_{\rm R}\,.$$

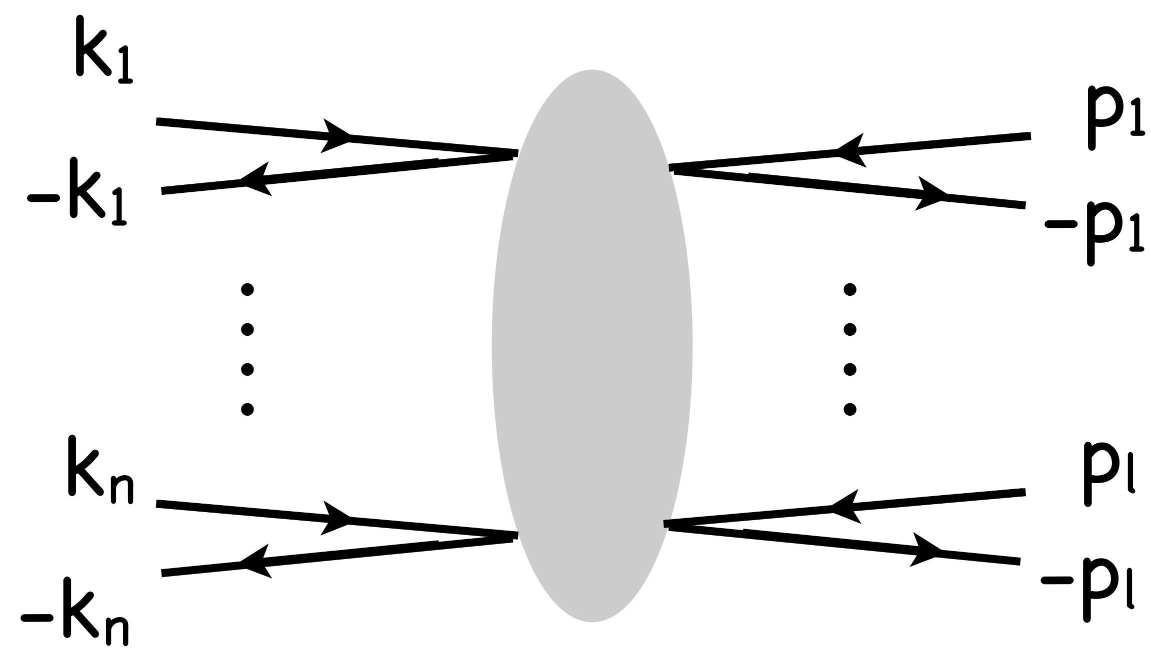
$$K(q) = \tan\left[\frac{\theta_{h'}(q)-\theta_h(q)}{2}\right]$$

4. Lehmann representation in terms of new Bogoliubov fermions

$$
{}_{\rm NS}\langle 0|\sigma^z_m(t)|0\rangle_{\rm R} = \sum_{l,n=0}^{\infty} \frac{1}{n!\,l!} \sum_{\substack{k_1,\ldots,k_n \\ p_1,\ldots,p_l}} \left[\prod_{j=1}^{n} K(k_j)\right] \left[\prod_{i=1}^{l} K(p_i)\right]
$$

$$
{}_{\rm NS}\langle -k_1, k_1, \ldots, -k_n, k_n|\sigma^z_m(t)|p_1, -p_1, \ldots, p_l, -p_l\rangle_{\rm R}
$$

$$
= \sum_{l,n=0}^{\infty} \frac{1}{n!\,l!}
$$



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$$

$$
\boxed{{}_{\mathrm{NS}}\langle -k_1, k_1, \dots, -k_n, k_n|\sigma_m^z(t)|p_1, -p_1, \dots, p_l, -p_l\rangle_{\mathrm{R}}}
$$

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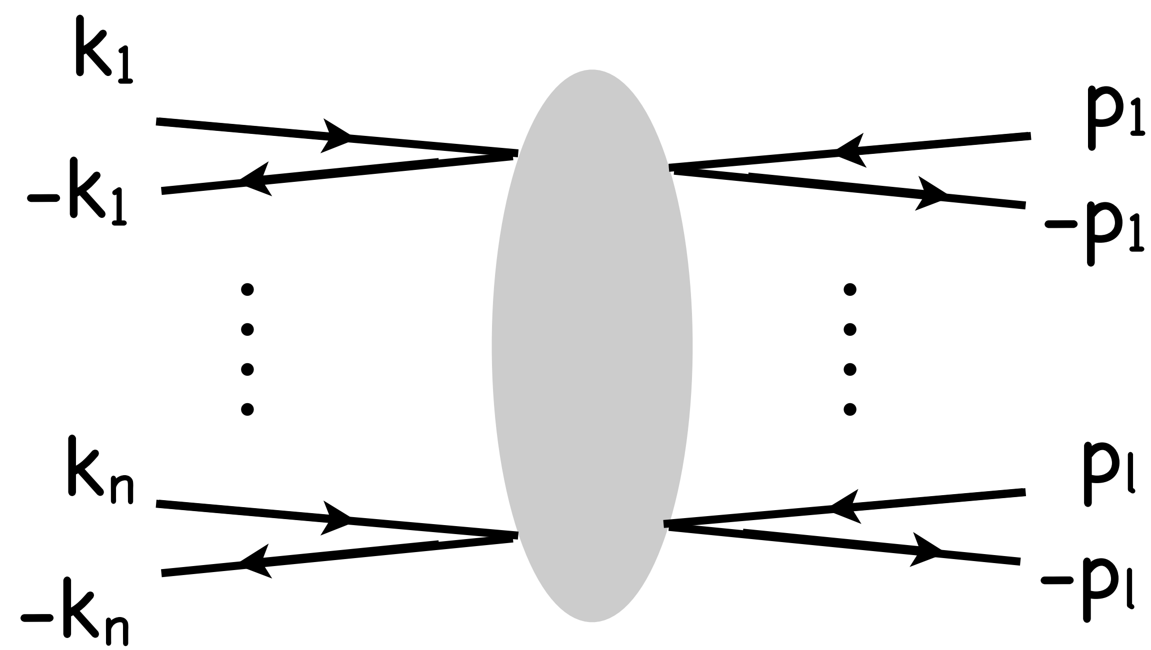

form factors are known exactly for the lattice model

(Vaidya&Tracy 1978, vonGehlen, Iorgov, Pakuliak, Shadura & Tykhyy 2008)

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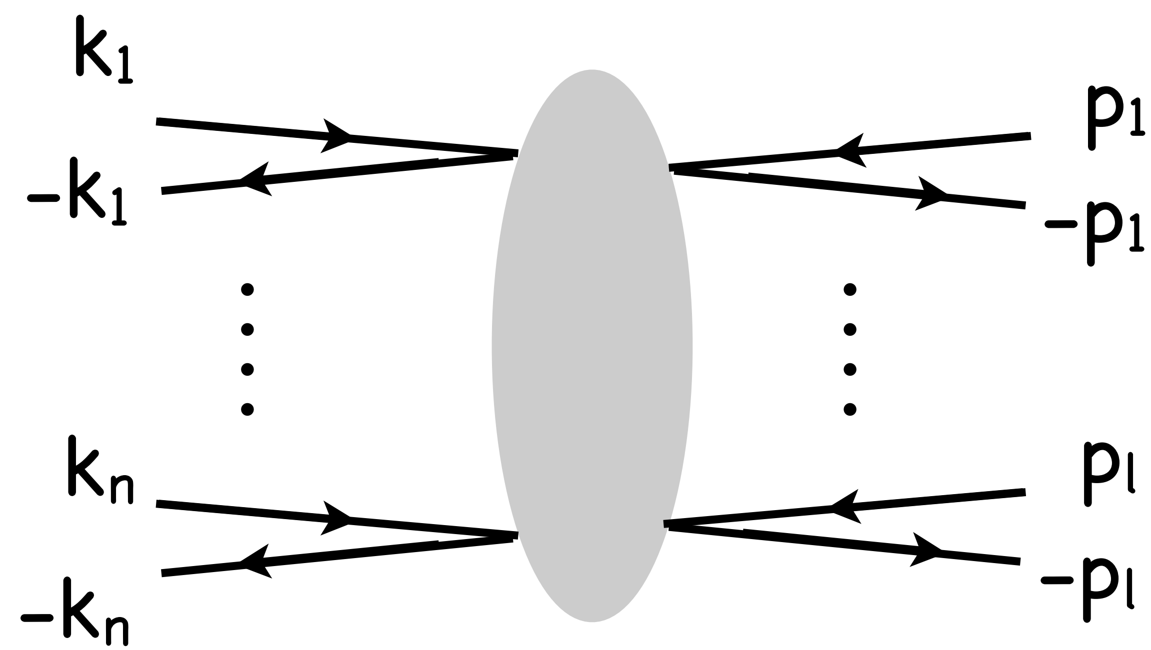

Idea: Consider K(q) as expansion parameter:

$$
\boxed{n(q) = \frac{\langle 0|\beta^\dagger_q \beta_q|0\rangle}{\langle 0|0\rangle} = \frac{K^2(q)}{1+K^2(q)}}
$$

density of excitations

n(q) small ⇔ K(q) uniformly small in q

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{}_{\mathrm{NS}}\langle 0|\sigma^z_m(t)|0\rangle_{\mathrm{R}} = \sum_{l,n=0}^{\infty} \frac{1}{n!\; l!} \sum_{\substack{k_1,\dots,k_n \\ p_1,\dots,p_l}} \left[\prod_{j=1}^{n} K(k_j)\right] \left[\prod_{i=1}^{l} K(p_i)\right]
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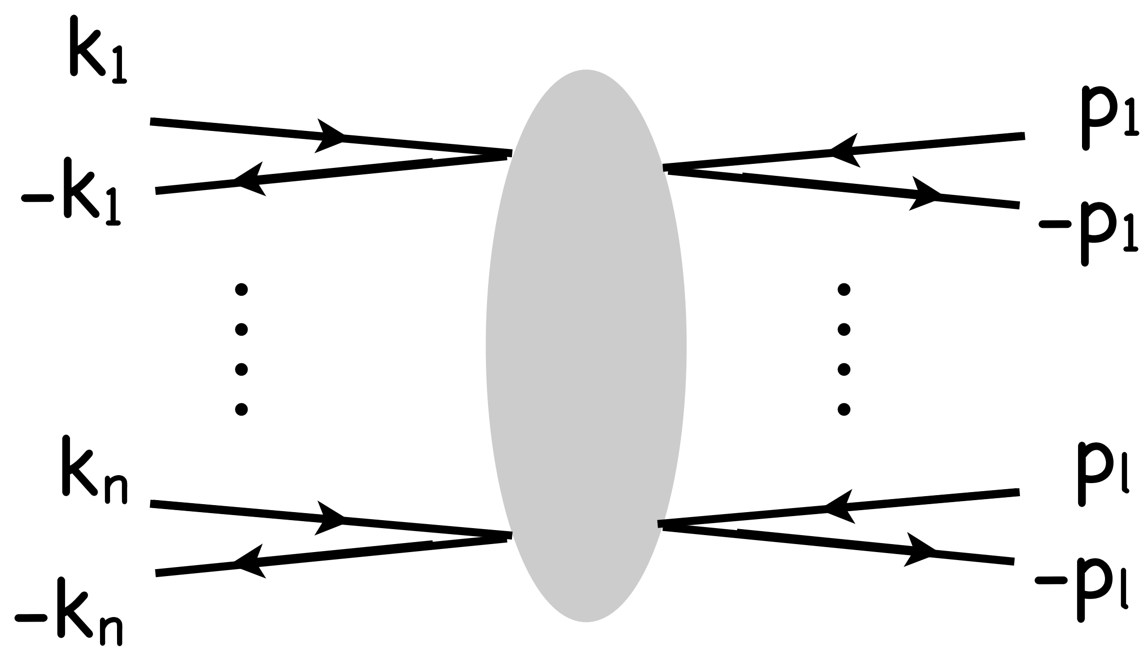

1. Dominant contributions from even orders $\mathbf{K^{2n}}$

2. Leading contributions **at order** $\mathbf{K^{2n}}$ from terms with $n=l$ and $\{k_1,\dots,k_n\} = \{p_1,\dots,p_n\}$

sum these to all orders $\Rightarrow$

$$\frac{\langle 0|\sigma_m^z(t)|0\rangle}{\langle 0|0\rangle} \propto \exp\left(-t\int_0^\pi \frac{dk}{\pi}\left[K^2(k)+\mathcal{O}(K^6)\right]|2\epsilon'(k)|\right)$$

- **Low density expansion** of the full answer.
- Works well everywhere except very close to QCP.
- 2-point function calculated similarly (more complicated).

Conclusions

1. Nonequilibrium evolution in integrable models appears to be special.

2. Nonlocality does not save the day. ☹

3. "Form factor" approach generalizes to "integrable" quenches (initial state ⇔ integrable boundary conditions)

⇒ mass quench ($m=\infty$ to m finite) in sine-Gordon

4. What happens for more general initial states (e.g. break translation invariance) ? ⇒ Ising chain.