Generalized quark-antiquark potential at weak and at strong coupling

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based on

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Outline

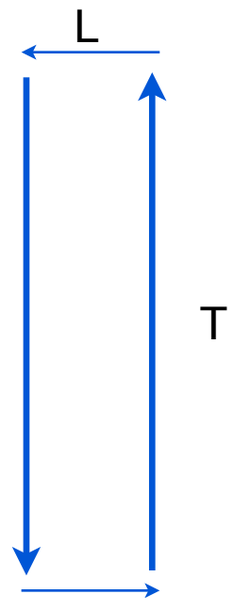
- Introduction and motivation
- Sketch of weak- and strong-coupling calculations
- Results & Remarks

Quark anti-quark potential in AdS/CFT (1)

- Potential between charged particles: fundamental observable in any qft.

  Gauge theory: *long* ($T \gg L$) rectangular Wilson loop, or a *pair of antiparallel lines.*

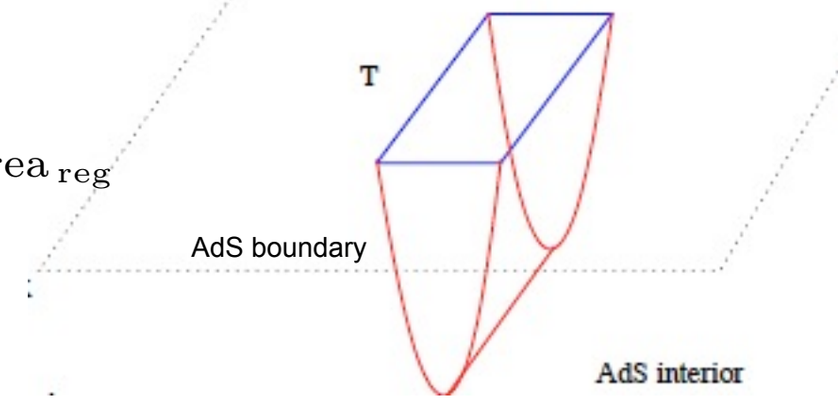
$$< W[\mathcal{C}] > \sim \; e^{-T\,V(L)}, \qquad\qquad T \to \infty$$



- *N=4 SYM*: potential between two infinitely massive W-bosons.

  String realization via AdS/CFT: effective action of a string ending on the curve at the boundary.

$$< W[\mathcal{C}] >= Z_{\rm string}[\mathcal{C}] = e^{-S_{\rm string}[\mathcal{C}]} \underset{\lambda \to \infty}{\longrightarrow} e^{-\sqrt{\lambda}\,{\rm Area}_{\rm reg}}$$



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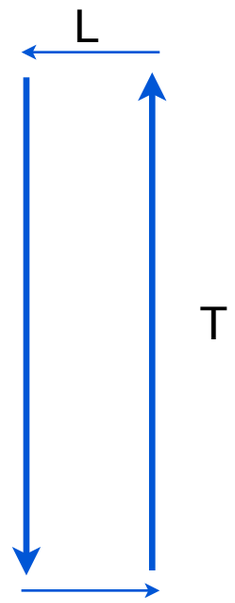

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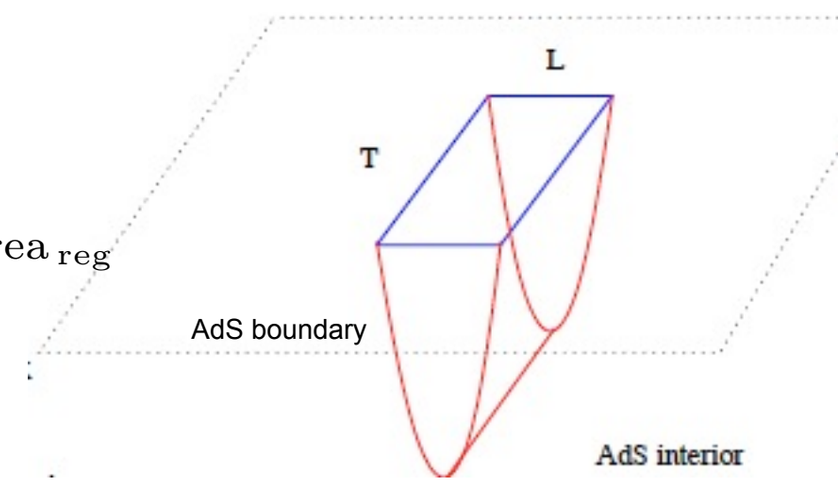

- For a conformal theory the potential is Coulomb-like

$$V(\lambda, L) \;=\; \frac{V(\lambda)}{L}$$

Is there a **way (integrability-related?)** to allow a calculation **at all values of the coupling** (in the planar limit)?

Quark anti-quark potential in AdS/CFT (2)

- Explicit calculations at weak and at strong coupling:

$$V(\lambda) = \begin{cases} -\frac{\lambda}{4\pi} + \frac{\lambda^2}{8\pi^3} \ln \frac{2\pi}{\lambda} + \ldots, & \lambda \ll 1 \\ -\frac{\sqrt{\lambda}\, 4\pi^2}{\Gamma^4(\frac{1}{4})} + \ldots, & \lambda \gg 1 \end{cases}$$

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- Weak coupling: could try 3 loops - hard perturbative calculation.
(Generalizing Smirnov² Steinhauser 2010 QCD-result?)

- Strong coupling: quantum ($\alpha' \sim \frac{1}{\sqrt{\lambda}}$) corrections are in general *hard* to evaluate, usually simplified by limits of scaling parameters (here *no* parameters).

First quantum correction: formally addressed 10-years ago, then numerical evaluation.

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We add extra parameters and study a larger family of observables. So to:

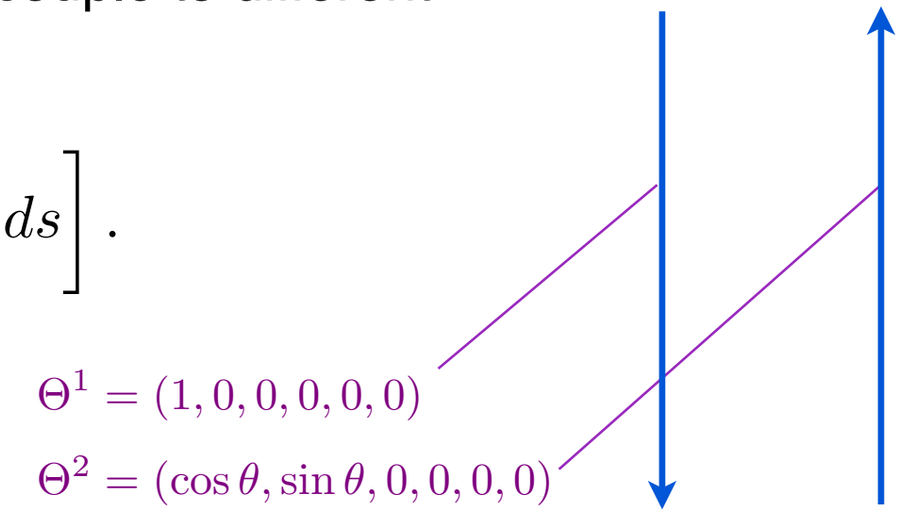
★ Build a data-set with explicit analytic formulas

★ Allow a clear connection with other, *protected*, observables.
Gather more information to help guessing an interpolating function.

Generalized quark-antiquark potential (1)

- First additional parameter (θ): allows the two lines to couple to different scalars.

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left[\oint (i A_\mu \dot{x}^\mu + \Phi_I \Theta^I |\dot{x}|) ds \right] .$$



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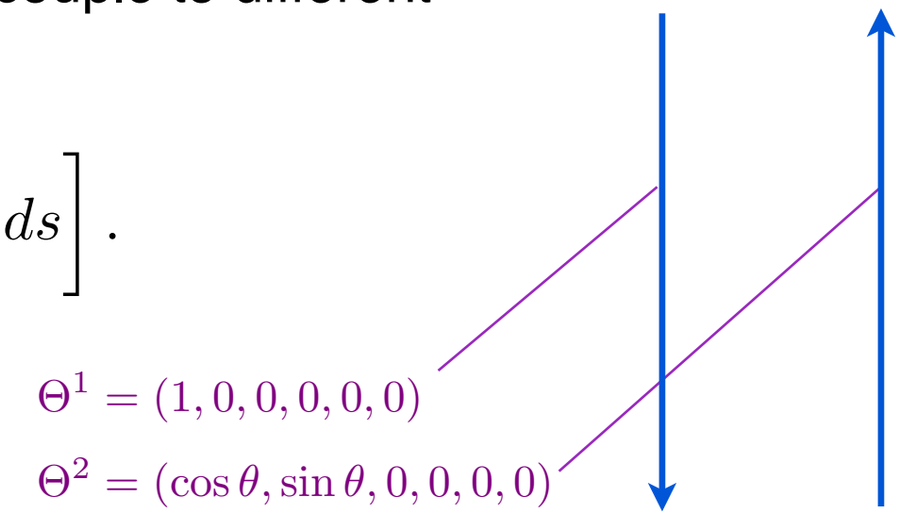
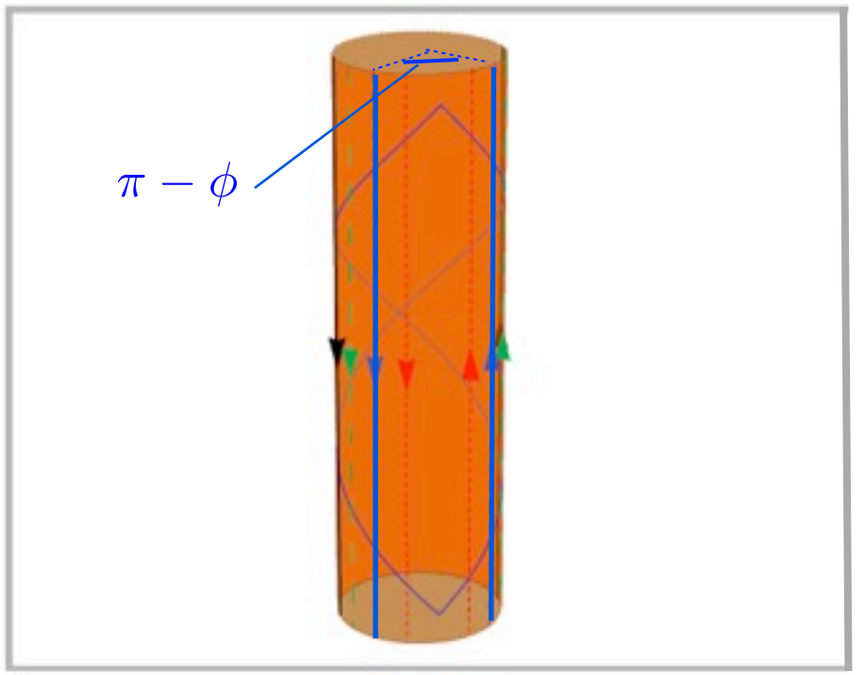


- Second additional parameter (ϕ): geometric, lines separated by $\pi - \phi$ in $\mathbb{R} \times S^3$

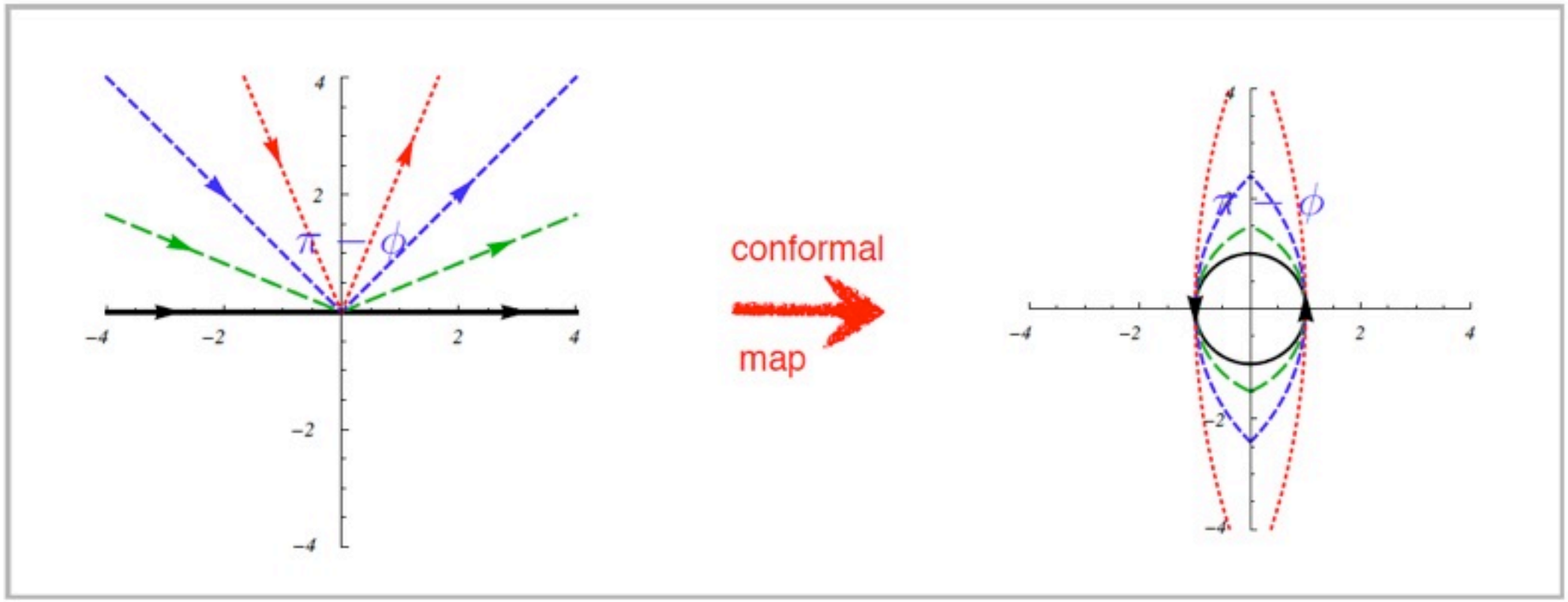


$\phi = 0$: BPS lines

$\phi \to \pi$: antiparallel lines

Generalized quark-antiquark potential (2)

- Wick-rotating to flat Euclidean space: cusp in $\mathbb{R}^4$



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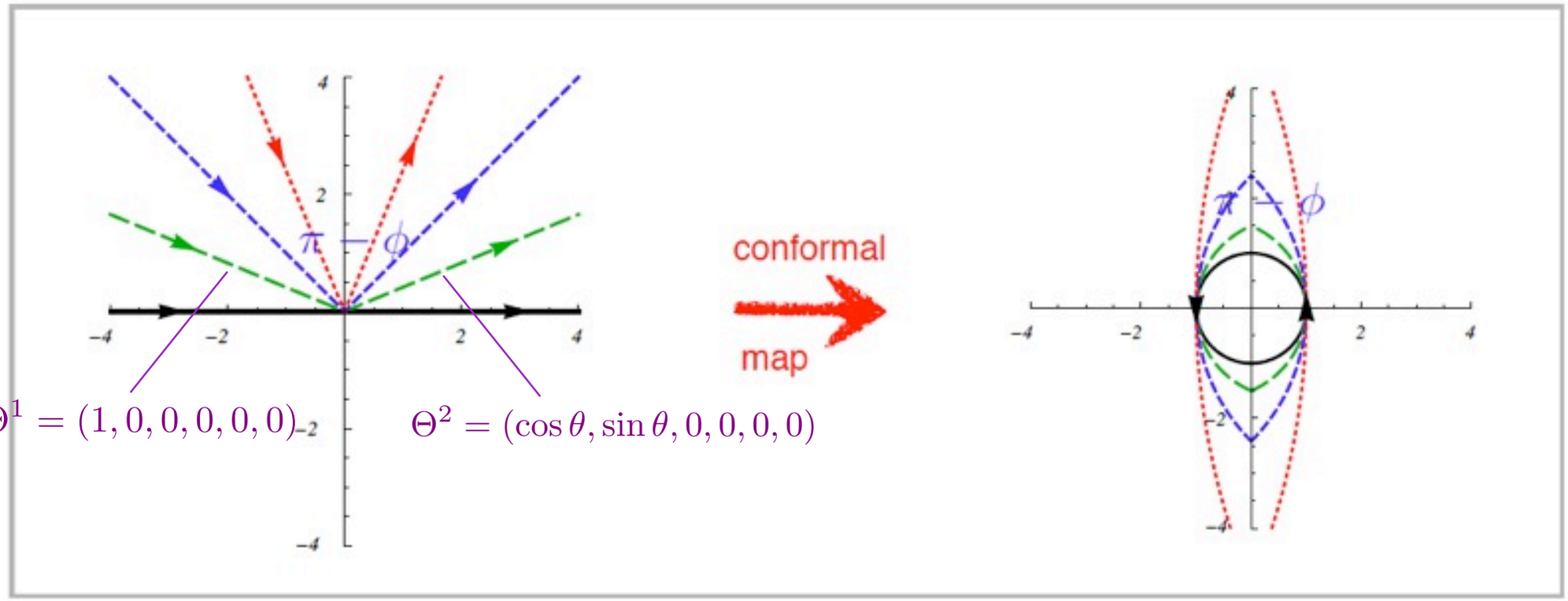

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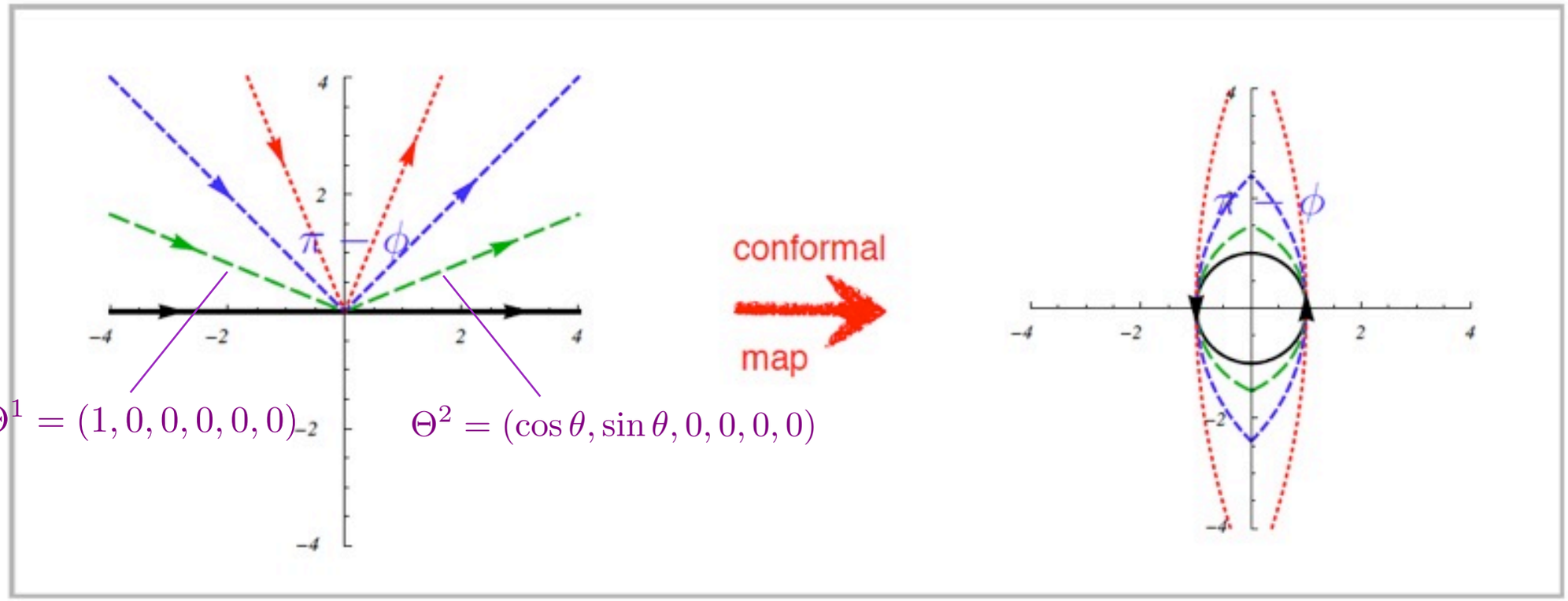

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- Cusped Wilson loops suffer *logarithmic* divergencies $\langle W \rangle \approx \exp[-\log \frac{R}{\epsilon}\, V(\phi, \theta, \lambda)]$

  Same as *linear* divergence of the antiparallel lines $\langle W \rangle \approx \exp[-T\, V(\phi, \theta, \lambda)]$

  if $\log \frac{R}{\epsilon} = T$.

Generalized quark-antiquark potential (3)

- $V(\phi, \theta, \lambda)$ is our generalization of $V(L, \lambda)$: *generalized quark-antiquark potential*.

  For $\phi \to \pi$ (and $\theta = 0$) it has a pole and the residue is $L\, V(L, \lambda)$.

- Expanding at weak coupling

$$V(\phi, \theta, \lambda) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{16\pi^2} \right)^n V^{(n)}(\phi, \theta) \ .$$

  We have evaluated the first two loops in perturbation theory.

- Expanding at strong coupling ($\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections)

$$V(\phi, \theta, \lambda) = \frac{\sqrt{\lambda}}{4\pi} \sum_{l=0}^{\infty} \left(\frac{4\pi}{\sqrt{\lambda}} \right)^l V_{AdS}^{(l)}(\phi, \theta) \, .$$

  We have evaluated the classical and 1-loop results in σ-model perturbation theory.

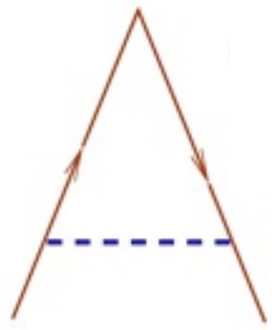
Weak-coupling calculation (1)

We worked with the cusp in $\mathbb{R}^4$.

- **1-loop**: just the exchange of a gluon and scalar field

[Drukker, Gross, Ooguri 1999]

$$V^{(1)}(\phi,\theta) = -2\,\frac{\cos\theta - \cos\phi}{\sin\phi}\,\phi$$



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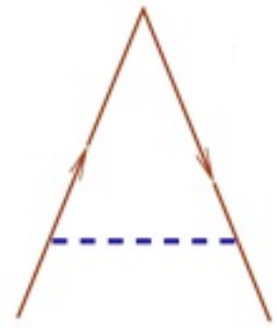


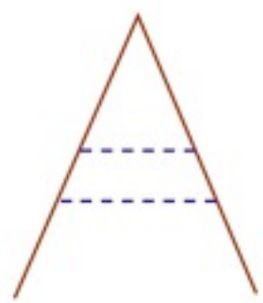
- **2-loops**: sum of ladder and interacting graphs

[Makeenko Olesen Semenoff 2006]

ϕ - result

$$V^{(2)}(\phi,\theta) = V^{(2)}_{\rm lad}(\phi,\theta) + V^{(2)}_{\rm int}(\phi,\theta)$$

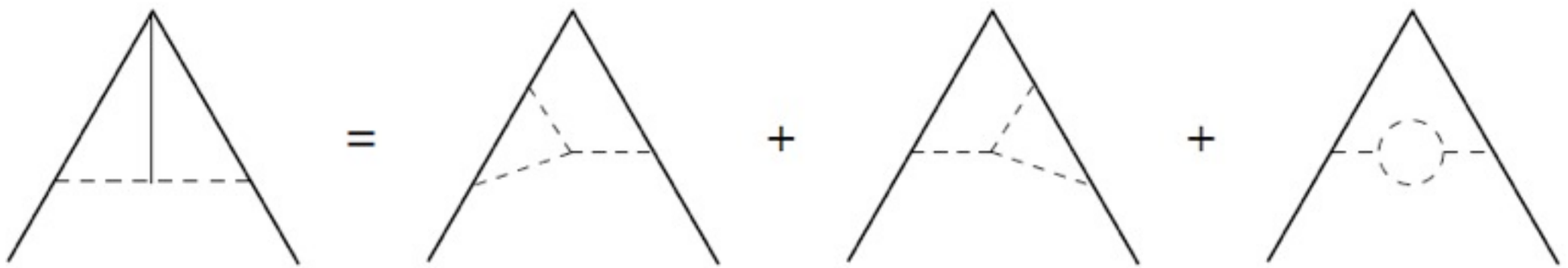
★ Ladder contribution



$$V^{(2)}_{\rm lad}(\phi,\theta) = -4\,\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\left[{\rm Li}_3\left(e^{2i\phi}\right) - \zeta(3) - i\phi\left({\rm Li}_2\left(e^{2i\phi}\right) + \frac{\pi^2}{6}\right) + \frac{i}{3}\phi^3\right]$$

Weak-coupling calculation (2)

★ **Interacting** contributions: graphs with *internal vertices* not cancelled by ***self-energy*** contributions as for smooth loops.



The triangle graph gives a result ***simpler*** than the ladder graphs

$$V_{\rm int}^{(2)}(\phi,\theta) = \frac{4}{3}\,\frac{\cos\theta-\cos\phi}{\sin\phi}\,(\pi^2-\phi^2)\phi$$

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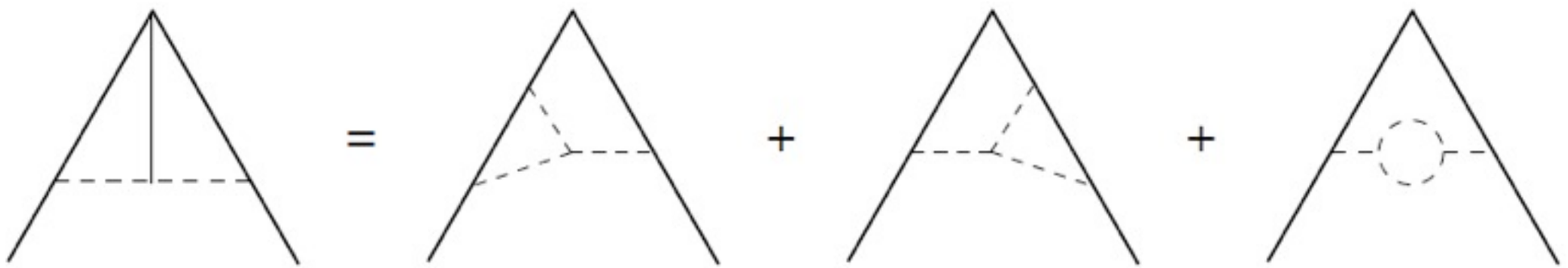

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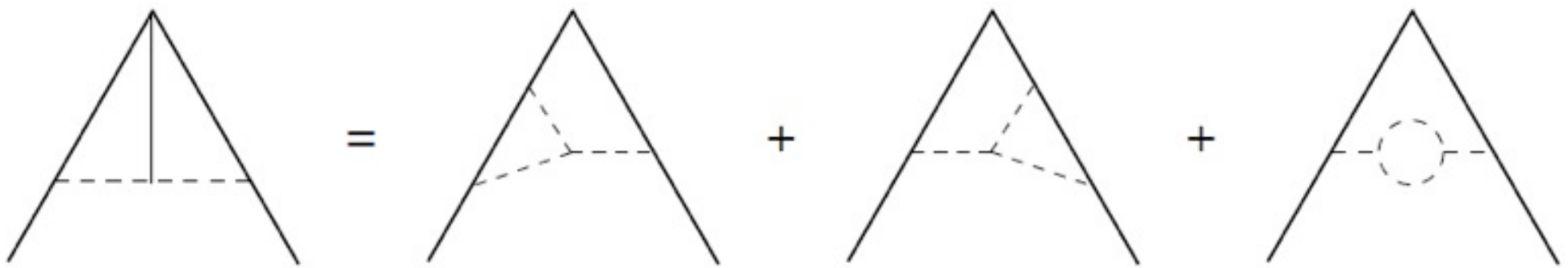

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- Checks:
  - ✔ BPS case ($\phi=\theta$): potential vanishes. [Drukker Giombi Ricci Trancanelli 2007]
  - ✔ Antiparallel lines limit ($\phi\to\pi$): replacing $\pi-\phi\to L$ gives the known result. [Erickson Semenoff Zarembo 1998]
  - ✔ Lorentzian null cusp ($\phi = i\,u$ and $u\to\infty$): matches anomalous dimension of twist-two operators with large spin, or cusp an. dim. of light-like Wilson loops. [Lipatov Kotikov 2000] [Korchemksky Radyushkin 1987]

Strong-coupling calculation: classical result (1)

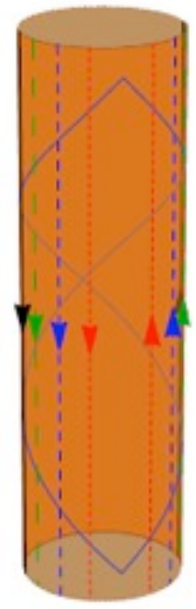
- It is sufficient to consider $AdS_3 \times S^1$

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2 + d\vartheta^2$$

- Take as world-sheet coordinates t and φ and

$$\rho = \rho(\varphi)\,, \qquad \vartheta = \vartheta(\varphi)$$

with ranges $\varphi \in [\frac{\phi}{2}, \pi - \frac{\phi}{2}]$ and $\vartheta \in [-\frac{\theta}{2}, \frac{\theta}{2}]$ (boundary at $\rho \to \infty$).



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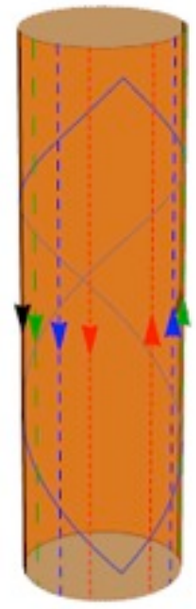

- The Nambu-Goto action gives the minimal area. Two conserved quantities (E, J).

The case $E = \pm J$ (or $\phi = \pm\theta$) is BPS. [Drukker Giombi Ricci Trancanelli 2007]

Equations of motion are *elliptic*, substituting the solution into the action gives

$$V^{(0)}_{AdS} = \frac{\mathcal{S}_{\rm cl}}{T} = \frac{\sqrt{\lambda}}{2\pi}\frac{2\sqrt{b^4+p^2}}{b\,p}\left[\frac{(b^2+1)p^2}{b^4+p^2}\mathbb{K}(k^2) - \mathbb{E}(k^2)\right].$$

where $p = 1/E,\ q = -J/E$ and

$$b^2 = \frac{p - q^2 + \sqrt{(p^2-q^2)^2 + 4p^2}}{2}, \quad k^2 = \frac{b^2(b^2-p^2)}{b^4+p^2}\ .$$

Transcendental equations for p, q in terms of θ, ϕ (2 parameters in the game).

Strong-coupling calculation: classical result (2)

✔ The limit $\phi \to \pi$ reproduces the original Maldacena result

$$\lim_{\phi\to\pi} V^{(0)}_{AdS}(\phi,\theta) = -\frac{4}{\sqrt{p}}\frac{\mathbb{E}-(1-k^2)\mathbb{K}}{\sqrt{k}(1-k^2)^{1/4}} = -\frac{8}{\pi-\phi}\frac{\left(\mathbb{E}-(1-k^2)\mathbb{K}\right)^2}{k\sqrt{1-k^2}}$$

with the identification (same at weak coupling) $\pi - \phi \to L$.

✔ Setting further $\theta = 0$ ($k^2 = 1/2$) one recovers the known result

$$V^{(0)}_{AdS}(\phi,0) = -\frac{16\pi^3}{(\pi-\phi)\Gamma\left(\frac{1}{4}\right)^4} \quad .$$

Strong-coupling calculation: 1-loop (1)

- Determinant of quantum fluctuations over the classical solution.

GS superstring action in $AdS_5 \times S^5$: $\quad I = -\dfrac{\sqrt{\lambda}}{2\,\pi} \int d^2\xi \left[L_B(x,y) + L_F(x,y,\theta) \right]$

$$L_B = \frac{1}{2}\sqrt{-g} g^{ab} \left[G^{(AdS_5)}_{mn}(x)\, \partial_a x^m \partial_b x^n + G^{(S^5)}_{m'n'}(y) \partial_a y^{m'} \partial_b y^{n'} \right]$$

$$L_F = i(\sqrt{-g} g^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ})\, \bar{\theta}^I\, \rho_a\, D_b\, \theta^J + \mathcal{O}(\theta^4)$$

[Metsaev, Tseytlin 98]

$$\frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'} = T$$

$$\hbar \leftrightarrow 1/\sqrt{\lambda}$$

Semiclassical expansion parameter

> Background field method $\qquad \mathcal{L} \xrightarrow{\phi = \phi_{cl} + \frac{\tilde{\phi}}{\sqrt[4]{\lambda}}} \tilde{\mathcal{L}}_{\rm fluct}$

> Effective action $\qquad \Gamma = -\ln Z = -\ln \dfrac{\det \text{fermions}}{\sqrt{\det \text{bosons}}} = -\ln < W >= -T\, V_{q\bar{q}}$

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- Use static gauge (fluctuations only trasverse to ws), rewrite metric in conformally flat form.

The result for *bosonic fluctuations* is

$$\mathcal{L}_B = \frac{1}{2}\sqrt{g}\Big[g^{ab}\, \partial_a \zeta_P\, \partial_b \zeta_P + M_{PQ} \zeta_P \zeta_Q \Big], \qquad P, Q = 1, \cdots, 8$$

with complicated mass matrix M_{PQ} , in general *non-diagonal*.

It becomes diagonal in the cases $(\phi, \theta = 0)$ and $(\phi = 0, \theta)$.

Strong-coupling calculation: 1-loop (2)

- Fermions: 8 species of Majorana fermions in 2 dimensions [Drukker,Gross, Tseytlin 00]

$$\mathcal{L}_{2F}^{\rm IIB} = -i\Big(\sqrt{g}\, g^{ij}\delta_{IJ} - \epsilon^{ij} s_{IJ}\Big)\bar{\psi}^I \gamma_i \Big(\delta^{IJ}\mathcal{D}_i - \frac{i}{2}\,\epsilon^{IJ}\,\Gamma_*\,\gamma_i\Big)\psi^J$$

$$\xrightarrow[\kappa\text{- symmetry fixing}]{\text{SO(1,9) rotation}} \quad \mathcal{L}_{2F} = \bar{\psi}\big(i\,\gamma^i\,\hat{\nabla}_i - M_F\big)\psi$$

M_F becomes diagonal in the cases $(\phi, \theta = 0)$ and $(\phi = 0, \theta)$.

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M_F becomes diagonal in the cases $(\phi, \theta = 0)$ and $(\phi = 0, \theta)$.

- Final form of the effective action (e.g. case $\theta = 0$)

$$\Gamma = -\ln \frac{\det^4(i\gamma^i\,\hat{\nabla}_i - \gamma_3)}{\det(-\nabla^2+2)\,\det^{1/2}(-\nabla^2 + R^{(2)} + 4)\,\det^{5/2}(-\nabla^2)}$$

> Invariance under time-translations ⟶ *1-dimensional determinants!*

$$\det\Big[-\partial_\tau^2 - \partial_\sigma^2 + M^2(\sigma)\Big] = \mathcal{T}\int\frac{d\omega}{2\pi}\Big[-\partial_\sigma^2 + \omega^2 + M^2(\sigma)\Big]$$

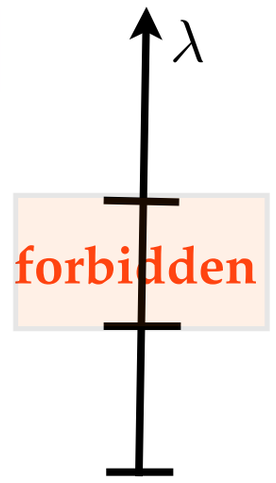
> *<u>But</u>* with non-trivial $M^2(\sigma)$, since $R^{(2)} = -2 - \dfrac{2k^2}{1-k^2}\,\mathrm{cn}^4(\sigma, k^2)$.

Lamé potential and Gelfand-Yaglom theorem

All fluctuations operators can be cast in the form of *Lamé operators*

$$\mathcal{O} = \sqrt{g}\left(-\nabla^2 + R^{(2)} + 4\right) = -\partial_\sigma^2 + \tilde{\omega}^2 + \frac{2(1-\tilde{k}^2)}{\mathrm{cn}^2(\sigma,\tilde{k}^2)} - 2\tilde{k}^2\,\mathrm{cn}^2(\sigma,\tilde{k}^2)$$

$$\longrightarrow \quad -\partial_x^2 + \omega^2 + 2\,k^2\;\mathrm{sn}^2(x|k^2)$$



> Band structure of the eigenvalue problem: single-gap operator.

> General solution of the Lamé equation

$$y_\pm(x) = \frac{H(x+\alpha)}{\Theta(x)}\,e^{\mp Z(\alpha)x}, \qquad \mathrm{sn}(\alpha,\tilde{k}^2) = \sqrt{1+\frac{1}{\tilde{k}^2}\left(1+\frac{\pi^2\,\omega^2}{4\,\mathbb{K}^2(\tilde{k}^2)}\right)}$$

Jacobi $Z,\ H,\ \Theta$ functions (known in terms of Jacobi -θ functions).

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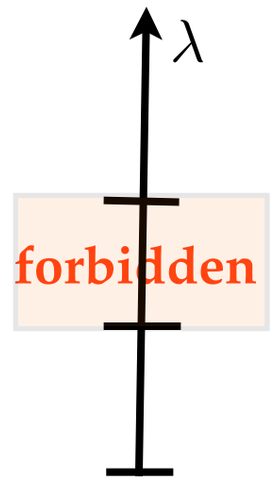

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Jacobi $Z,\ H,\ \Theta$ functions (known in terms of Jacobi -θ functions).

- Use Gelfand-Yaglom theorem = solve initial value problem with b.c. in x∈[0,L]

  $\longrightarrow$ determinant = {solution of associated homog. equation}$_{x=L}$

- Take solution satisfying $\quad u(-\mathbb{K}) = 0, \quad u'(-\mathbb{K}) = 1$

  $\longrightarrow \quad \det\mathcal{O} = u(\mathbb{K})$

- Need to regularize at the boundary: interval with cutoff, $\sigma \in [-\mathbb{K}+\epsilon, \mathbb{K}-\epsilon]$.

Regularized effective action & 1-loop potential

- Explicit expression of the reg. determinant

$$\det \widetilde{\mathcal{O}}_2 = -\frac{\omega^2 \mathrm{dn}(\epsilon|k^2)^2 - k^2 + 1}{\omega\sqrt{(\omega^2+1)(-k^2+\omega^2+1)}\ \mathrm{dn}(\epsilon|k^2)^2} \sinh\left(2Z(\widetilde{\alpha}_2)(\mathbb{K}-\epsilon) + \widetilde{\Sigma}_2\right)$$

where

$$\mathrm{sn}(\widetilde{\alpha}_2|k^2) = \sqrt{\frac{1+\omega^2}{k^2}}\,, \qquad \Sigma_2 = \ln \frac{\vartheta_2\left(\frac{\pi(\epsilon+\bar{\alpha}_2)}{2\mathbb{K}}, q\right)}{\vartheta_2\left(\frac{\pi(\epsilon-\bar{\alpha}_2)}{2\mathbb{K}}, q\right)}$$

Regularized effective action & 1-loop potential

- Explicit expression of the reg. determinant

$$\det \widetilde{\mathcal{O}}_2^{\epsilon} \cong -\frac{\sinh(2\mathbb{K}_2\, Z(\alpha_2))}{\epsilon^2\, \omega \sqrt{\omega^4 + (2-4k^2)\omega^2 + 1}}$$

where

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What counts is leading term in the ϵ - expansion.

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What counts is leading term in the ϵ - expansion.

- The regularized effective action reads then

$$\Gamma_{\rm reg}=-\frac{\mathcal{T}}{2}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln\frac{\epsilon^2\omega^2\det{}^8\,\widetilde{\mathcal{O}}_F^{\epsilon}}{\det{}^4\,\widetilde{\mathcal{O}}_0^{\epsilon}\det{}^3\,\widetilde{\mathcal{O}}_1^{\epsilon}\det\,\widetilde{\mathcal{O}}_2^{\epsilon}}\,.$$

from which read off the 1-loop correction to the generalized potential

$$V_{AdS}^{(1)}(\phi,0)=\frac{1}{T}\Gamma_{\rm reg}$$

Similarly one gets $V_{AdS}^{(1)}(0,\theta)$.

Results (1)

Explicit formulas for these families of Wilson loops at weak and at strong coupling.

λ

$$V^{(1)} = -2\,\frac{\cos\theta - \cos\phi}{\sin\phi}\,\phi$$

$$V^{(2)} = V^{(2)}_{\rm lad} + V^{(2)}_{\rm int}$$

$$V^{(2)}_{\rm lad} = -4\,\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\left[{\rm Li}_3\left(e^{2i\phi}\right) - \zeta(3) - i\phi\left({\rm Li}_2\left(e^{2i\phi}\right) + \frac{\pi^2}{6}\right) + \frac{i}{3}\phi^3\right],$$

$$V^{(2)}_{\rm int} = \frac{4}{3}\,\frac{\cos\theta-\cos\phi}{\sin\phi}\,(\pi-\phi)(\pi+\phi)\phi$$

$$\vdots$$

$$V^{(1)}_{AdS} = -\frac{\mathcal{T}}{2\,T}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln\frac{\epsilon^2\omega^2\det{}^8\,\widetilde{\mathcal{O}}^\epsilon_F}{\det{}^4\,\widetilde{\mathcal{O}}^\epsilon_0\,\det{}^3\,\widetilde{\mathcal{O}}^\epsilon_1\,\det\,\widetilde{\mathcal{O}}^\epsilon_2}$$

$$\det\mathcal{O}^\epsilon_0 \cong \frac{\sinh(2\mathbb{K}\,\omega)}{\omega}\,,$$

$$\det\mathcal{O}^\epsilon_1 \cong -\frac{\sinh(2\mathbb{K}_1\,Z(\alpha_1))}{\epsilon^2\,\sqrt{k^2-\omega^2}\,\sqrt{(\omega^2-k^2+1)(-\omega^2+2k^2-1)}}\,,$$

$$\det\mathcal{O}^\epsilon_2 \cong -\frac{\sinh(2\mathbb{K}_2\,Z(\alpha_2))}{\epsilon^2\,\omega\sqrt{\omega^4+(2-4k^2)\omega^2+1}}\,,$$

$$\det\mathcal{O}^\epsilon_F \cong \frac{8\mathbb{K}_2\sqrt{\omega_3^2+k_2^2}\,\sinh(\mathbb{K}_2\,Z(\alpha_F))}{\epsilon\pi(1-k^2)(k_1+1)^2\sqrt{(\omega_3^2+1)(\omega_3^2+k_2^2+1)}}\,\frac{\vartheta_2(0,q_2)\,\vartheta_4\left(\frac{\pi\alpha_F}{2\mathbb{K}_2},q_2\right)}{\vartheta_1'(0,q_2)\,\vartheta_3\left(\frac{\pi\alpha_F}{2\mathbb{K}_2},q_2\right)}$$

$$V^{(0)}_{AdS} = \frac{\sqrt{\lambda}}{2\pi}\,\frac{2\sqrt{b^4+p^2}}{b\,p}\left[\frac{(b^2+1)p^2}{b^4+p^2}\mathbb{K}(k^2) - \mathbb{E}(k^2)\right].$$

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⋮

Numerically evaluable with high precision

$$V^{(1)}_{AdS} = -\frac{\mathcal{T}}{2\,T}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln\frac{\epsilon^2\omega^2\det{}^8\,\widetilde{\mathcal{O}}^\epsilon_F}{\det{}^4\,\widetilde{\mathcal{O}}^\epsilon_0\,\det{}^3\,\widetilde{\mathcal{O}}^\epsilon_1\,\det\,\widetilde{\mathcal{O}}^\epsilon_2}$$

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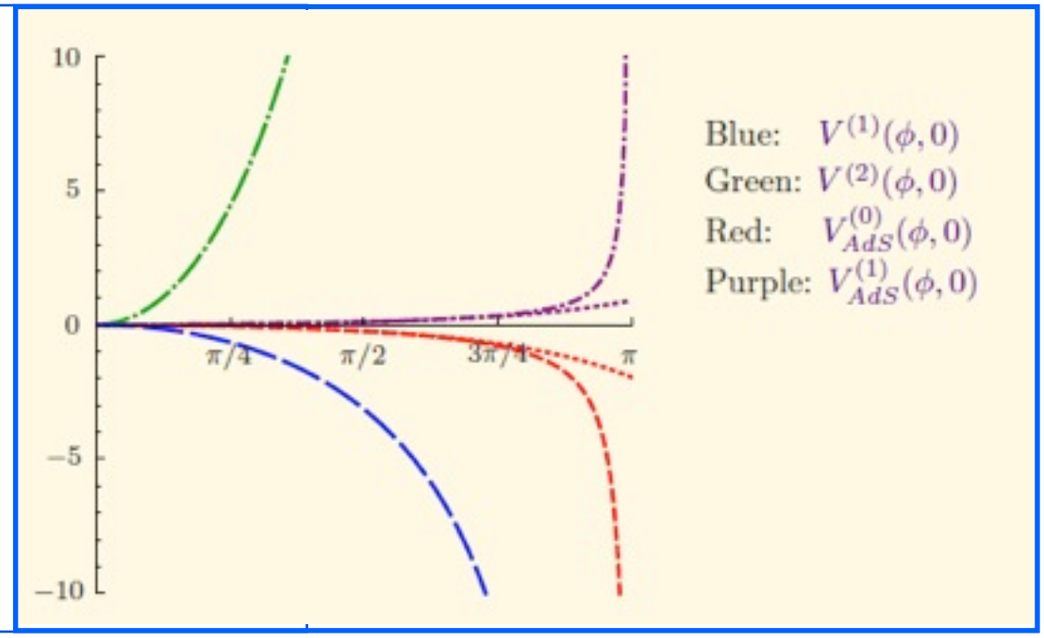

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Results (2)

Explicit analytic expressions for expansion around BPS (straight line)

- At weak coupling

$$V^{(1)}(\phi,\theta) = \theta^2 - \phi^2 - \frac{1}{12}(\theta^2-\phi^2)^2 + \frac{1}{360}(\theta^2-\phi^2)^2(\theta^2-3\phi^2) + O((\phi,\theta)^8)\,,$$

$$\begin{aligned} V^{(2)}(\phi,\theta) = & -\frac{2\pi^2}{3}(\theta^2-\phi^2) + \frac{1}{18}(\pi^2(\theta^2-\phi^2)^2 + 6(\theta^2-\phi^2)(3\theta^2-\phi^2)) \\ & - \frac{1}{540}(\pi^2(\theta^2-\phi^2)^2(\theta^2-3\phi^2) + 30(\theta^2-\phi^2)^2(3\theta^2-\phi^2)) + O((\phi,\theta)^8) \end{aligned}$$

- At strong coupling

$$\begin{aligned} V^{(0)}_{AdS}(\phi,\theta) = & \frac{1}{\pi}(\theta^2-\phi^2) - \frac{1}{8\pi^3}(\theta^2-\phi^2)\left(\theta^2-5\phi^2\right) + \frac{1}{64\pi^5}(\theta^2-\phi^2)\left(\theta^4-14\theta^2\phi^2+37\phi^4\right) \\ & - \frac{1}{2048\pi^7}(\theta^2-\phi^2)\left(\theta^6-27\theta^4\phi^2+291\theta^2\phi^4-585\phi^6\right) + O((\phi,\theta)^{10})\,. \end{aligned}$$

$$\begin{aligned} V^{(1)}_{AdS}(\phi,0) = & \frac{3}{2}\frac{\phi^2}{4\pi^2} + \left(\frac{53}{8} - 3\,\zeta(3)\right)\frac{\phi^4}{16\pi^4} + \left(\frac{223}{8} - \frac{15}{2}\zeta(3) - \frac{15}{2}\zeta(5)\right)\frac{\phi^6}{64\pi^6} \\ & + \left(\frac{14645}{128} - \frac{229}{8}\zeta(3) - \frac{55}{4}\zeta(5) - \frac{315}{16}\zeta(7)\right)\frac{\phi^8}{256\pi^8} + O(\phi^{10})\,, \\ V^{(1)}_{AdS}(0,\theta) = & -\frac{3}{2}\frac{\theta^2}{4\pi^2} + \left(\frac{5}{8} - 3\,\zeta(3)\right)\frac{\theta^4}{16\pi^4} + \left(\frac{1}{8} + \frac{3}{2}\zeta(3) - \frac{15}{2}\zeta(5)\right)\frac{\theta^6}{64\pi^6} \\ & + \left(-\frac{11}{128} - \frac{5}{8}\zeta(3) + \frac{25}{4}\zeta(5) - \frac{315}{16}\zeta(7)\right)\frac{\theta^8}{256\pi^8} + O(\theta^{10})\,. \end{aligned}$$

Expansion around BPS configuration (1)

- Focus on the first expansion coefficients around $\theta = \phi = 0$.

$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0} = -\frac{1}{2}\frac{\partial^2}{\partial\phi^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0} = \begin{cases} \dfrac{\lambda}{16\pi^2} - \dfrac{\lambda^2}{384\pi^2} + \cdots & \lambda \ll 1\,, \\ \dfrac{\sqrt{\lambda}}{4\pi^2} - \dfrac{3}{8\pi^2} + \cdots & \lambda \gg 1\,. \end{cases}$$

- Can be derived from straight Wilson loop operator with operator insertions

$$\frac{\partial^2}{\partial\theta^2}V(0,0) = -\frac{1}{T}\frac{\partial^2}{\partial\theta^2}\log\langle W\rangle \approx -\frac{1}{T}\frac{\partial^2}{\partial\theta^2}\langle W\rangle.$$

- Write the Wilson loop as

$$W = \mathrm{Tr}\,\mathcal{P}\Big[\exp\Big(\int_{-\infty}^{0}(iA_1+\Phi_1)ds\Big)\exp\Big(\int_{-\infty}^{0}(iA_1+\Phi_1\cos\theta+\Phi_2\sin\theta)ds\Big)\Big]$$

- The variation gives

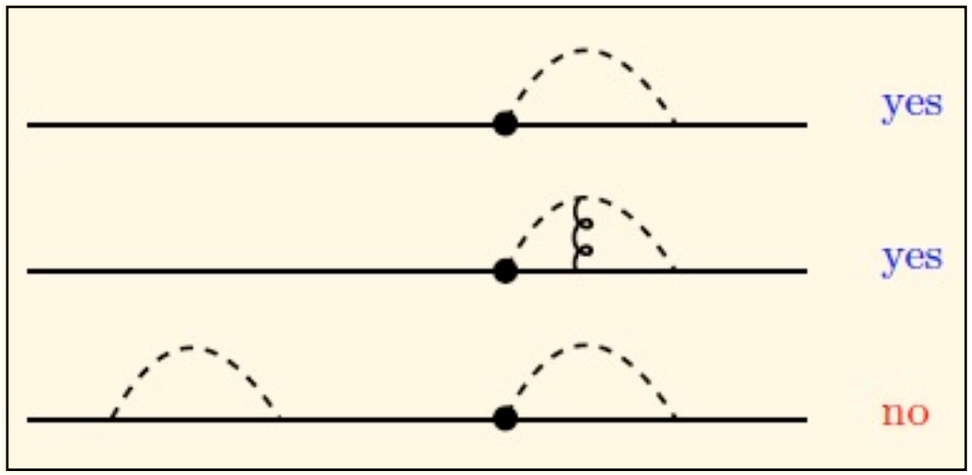
$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V = -\frac{1}{\ln(R/\epsilon)}\frac{1}{2N}\int_0^\infty ds_1\int_0^\infty ds_2\left\langle \mathrm{Tr}\,\mathcal{P}\left[\Phi_2(s_1)\Phi_2(s_2)\,e^{\int_{-\infty}^{\infty}(iA_1+\Phi_1)ds}\right]\right\rangle$$

$$+\frac{1}{\ln(R/\epsilon)}\frac{1}{2N}\int_0^\infty ds_1\left\langle \mathrm{Tr}\,\mathcal{P}\left[\Phi_1(s_1)\,e^{\int_{-\infty}^{\infty}(iA_1+\Phi_1)ds}\right]\right\rangle.$$

- The double insertion gives no log divergences, and is related to a BPS quantity.

Expansion around BPS configuration (2)

- To the single insertion some graphs will contribute and some not:



Only the “most interacting” graphs should contribute.

- Indeed the 2-loop ladder graphs contributes only from $\mathcal{O}((\theta,\phi)^4)$.

  The connected 2-loop graphs were also the simpler ones (did not include polylogs)..

Investigation of this subset of graphs looks promising.

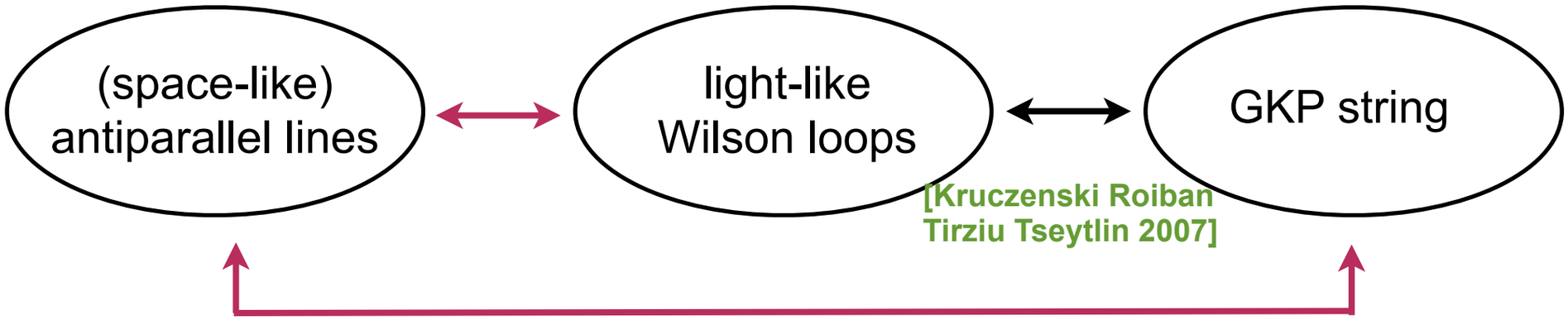
Concluding remarks (1)

- Analytically continuing $u = i\,\phi$ one obtains loops in Minkowski.
  - ✔ In the limit $u \to \infty$ cusp anomalous dimension reproduced at weak coupling.

  Cusp anomalous dimension: crucial for scattering amplitudes in N=4 SYM.

- Away from the limit, our results could be useful for generalizations or regularization of amplitudes.

- Interesting to work out <u>explicit</u> relation to GKP string solution.
  Step toward the use of integrability techniques for evaluation of Wilson loops!



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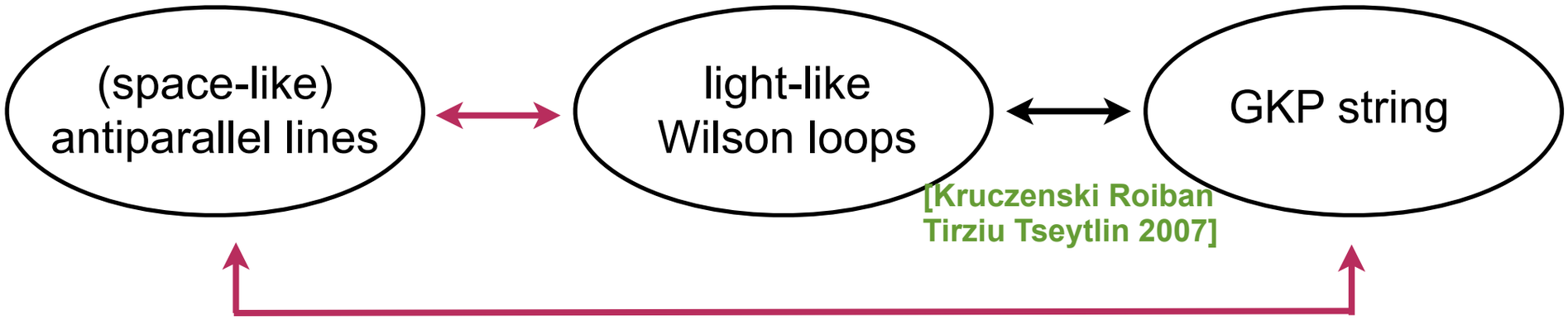

- Semiclassical fluctuations governed by *<u>Lamé</u>* operators (also folded string solution, also in $AdS_4 \times \mathbb{C}P^3$). *<u>Integrable</u>* differential operators.

  Finite gap operators: very well known Riemann surface and elliptic curve interpretation
  ⟶ connection to algebraic approach to integrability.

  ✎ Detailed map between qft- and algebraic curve- approach.
  Nice recent step by Kruczenski et al. (classical level): explore further.

Concluding remarks (2)

- Here, at strong coupling, two 1-parameter families of Wilson loop, $(\phi, \theta = 0)$ and $(\phi = 0, \theta)$

  - General(ϕ, θ) case ⟶ <u>coupled</u> fluctuations system.
    Spectrum = Solution of complicated matrix differential equations.

    Systematize knowledge of σ-model pert. theory in curved backgrounds
    in the semiclassical approx (solutions with many "spins", no simplifying limits).

    Classification of the DE associated to small fluctuations.

    Work in progress

- Here simplest generalization of antiparallel lines geometry.

  - Other deformation possible: ex. helical curves in flat space or $S^3 \times \mathbb{R}$ with extra-rotations around S^5.

- Exploit similar analytic methods in a Schwarzschild-AdS_5 background: possible, but no integrable operators appear. Numerics in Chu et al. 2011.

Concluding remarks (3)

- *Generalized quark-antiquark potential*: family of Wilson loops interpolating between 1/2 BPS line and parallel lines.

  - ✔ Weak-coupling results: two-loop order.
  - ✔ Strong-coupling results: one-loop order in σ-model pert. theory.

- Expanding around 1/2 BPS line: *analytic* results at weak and at strong coupling.

- Example of a quantity which should receive contributions only from a subset of graphs in perturbation theory.

$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0} = -\frac{1}{2}\frac{\partial^2}{\partial\phi^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0} = \begin{cases} \dfrac{\lambda}{16\pi^2} - \dfrac{\lambda^2}{384\pi^2} + \cdots & \lambda \ll 1\,, \\ \dfrac{\sqrt{\lambda}}{4\pi^2} - \dfrac{3}{8\pi^2} + \cdots & \lambda \gg 1\,. \end{cases}$$

Concluding remarks (3)

What will lead to a gauge theory derivation of the strong coupling potential?

$$V(L,\lambda) = \frac{4\,\pi^2\,\sqrt{\lambda}}{\Gamma(\frac{1}{4})^4\,L}$$

Grazie.