8th Bologna Workshop on:

CFT AND INTEGRABLE MODELS

Unusual singular behavior of the entanglement entropy in one dimension

ma per seguir virtute e canoscenza

SISSA

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- JPA 40: 8467 (2007)
- JPA 41: 2530 (2008)
- Quant. Inf. Proc. 10: 325 (2011)
- PRB 83: 12402 (2011)
- work in progress…

Conclusions

- Entanglement Entropy: non-local correlator
- Conformal prediction: $S \simeq \frac{c}{3} \ln \xi + \dots$

  → Due to simple scaling laws
- Close to non-conformal points: competition between different length scales → essential singularity
- Similar to partition function, ground state energy, but simpler…
- Entropy to distinguish relativistic vs. Galilean spectrum

Outline

- Introduction: Von Neumann and Renyi Entropy as a measure of Entanglement
- Entanglement Entropy in 1-D systems
- XY in magnetic field & XYZ in zero field
- Essential Critical Point for the entropy
- Quantum Entropy in integrable models
- Conclusions

Introduction

- Entanglement: fundamental quantum property
- Different reasons for interest:
  1. Quantum information $\rightarrow$ quantum computers
  2. Quantum Phase Transitions $\rightarrow$ universality
  3. Condensed matter $\rightarrow$ non-local correlator
  4. Integrable Models $\rightarrow$ new playground
  5. Cosmology $\rightarrow$ Black Holes
  6. …

Understanding Entanglement

- Consider a unique (pure) ground state

- Divide the system into two Subsystems: A & B

- If system wave-function is:

$$|\Psi^{A,B}\rangle = |\Psi^A\rangle \otimes |\Psi^B\rangle$$

$\rightarrow$ No Entanglement

(Measurements on B do not affect A state)

Understanding Entanglement (cont.)

- If the system wave-function is:

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{d} \lambda_j |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle$$

(with $d > 1$, $|\Psi_j^A\rangle$ & $|\Psi_j^B\rangle$ linearly independent):

$\rightarrow$ Entangled (Measurements on B affect A state):

i.e. $\langle\Psi_i^B|\Psi^{A,B}\rangle = \lambda_j|\Psi_i^A\rangle$

Simple Example

- Two spins 1/2 in triplet state $\rightarrow S_z = 1$:

$$|\uparrow\rangle \otimes |\uparrow\rangle$$

No entanglement

- Middle component with $S_z = 0$:

$$|\uparrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle$$

Maximally entangled

Entanglement Entropy

- Compute Density Matrix of subsystem:

$$\rho_A = tr_B \left(|\Psi^{A,B}\rangle\langle\Psi^{A,B}| \right)$$

- Entanglement for pure state as Quantum Entropy (Bennett, Bernstein, Popescu, Schumacher 1996):

$$S = -tr_A \left(\rho_A \ln \rho_A \right)$$

Von Neumann Entropy

- Ubiquitous in many areas of physics (cosmology, critical phenomena, strongly corr.)

More Entanglement Estimators

$$\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}|$$

- Von Neumann Entropy: $S_A = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$
- Renyi Entropy: $S_\alpha = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$

  (equal to Von Neumann for $\alpha \to 1$)
- Tsallis Entropy
- Concurrence (Two-Tangle)
- ...

The Von Neumann Entropy

$$S_A = -\mathrm{tr}_A\left(\rho_A \log \rho_A\right)$$

- Quantum analog of Shannon Entropy:

$$\rho_A = \sum \lambda_j |\Psi_j^A\rangle \langle \Psi_j^A| \quad \rightarrow \quad S_A = -\sum \lambda_j \log \lambda_j$$

- Measures the amount of "information" in the given state

- Remark: $S_B = -\mathrm{tr}_B\left(\rho_B \log \rho_B\right) = S_A$

Entropy as a measure of entanglement

- Assume Bell State as unity of Entanglement:

$$|\mathrm{Bell}\rangle = \frac{|\downarrow\downarrow\rangle \pm |\uparrow\uparrow\rangle}{\sqrt{2}}, \frac{|\downarrow\uparrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}}$$

- Von Neumann Entropy measures how many Bell-Pairs are contained in a given state $|\Psi^A\rangle$ (i.e. closeness of state to maximally entangled one)

Entanglement in a Spin Chain

- Consider the Ground state of a Hamiltonian:

$$H = \sum_{i=1}^{N} \left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right] - h \sum_i \sigma_i^z$$

- Block of spins in the space interval [1, n] is subsystem A
- The rest of the ground state is subsystem B.

→ Entanglement of a block of spins in the space interval [1, n] with the rest of the ground state as a function of n

Entropy as a Correlation Function

- Bi-partite entropy of the ground state of a system:

$$\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle \langle\Psi^{A,B}|$$

$$S(n) = -\mathrm{tr}\left(\rho_A \ln \rho_A\right)$$ Von Neumann

$$S_\alpha(n) = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$$ Renyi

- Multi-Point (n) correlation function with contributions from every 2-point correlators
- Highly non-trivial correlation function: new insights?

General Behavior (Area Law)

- Asymptotic behavior (block size $n \to \infty$)

  (Double scaling limit: $0 << n << N$)

$$S(n) = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$

- For gapped phases: (Vidal, Latorre, Rico, Kitaev 2003)

$$S(n) \simeq \mathrm{Constant} + \ldots$$

- For critical conformal phases: (Calabrese, Cardy 2004)

$$S(n) \simeq \frac{c}{3} \ln n + \ldots$$

Entropy and Universality (CFT)

$$S_\alpha(n) = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$$

- Powers of ρ easily accessible in CFT (replica)

(Cardy, Calabrese 2010)

$$S_\alpha(n) \simeq \frac{c}{6}\left(1+\frac{1}{\alpha}\right)\ln n + c'_\alpha + b_\alpha n^{-2x/\alpha} + \ldots$$

- Close to criticality: $\xi \sim \Delta^{-1}, n \to \infty$

(Calabrese, Cardy, Peschel 2010)

$$S_\alpha \simeq \frac{c}{6}\left(1+\frac{1}{\alpha}\right)\ln \xi + C'_\alpha + B_\alpha \xi^{-2x/\alpha} + \ldots$$

Conjectured

Entanglement and Integrability

$$H=\sum_{i=1}^{N}\left[J_x\sigma_i^x\sigma_{i+1}^x+J_y\sigma_i^y\sigma_{i+1}^y+J_z\sigma_i^z\sigma_{i+1}^z\right]-h\sum_i\sigma_i^z$$

- Take two integrable chains

1) XY in transverse field ($J_z = 0$)

2) XYZ in zero field ($h = 0$)

- We study the entanglement in the $n \rightarrow \infty$ limit

→ **entanglement of one half-line with the other**

The Anisotropic XY Model

$$\mathbf{H} = -\sum_{i}\left[(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y + h\,\sigma_i^z\right]$$

$\downarrow$

Exact (non-local) mapping into free fermions

(Jordan-Wigner + Bogolioubov rotation)

$\downarrow$

$$\mathbf{H} = \sum_{q}\varepsilon_q\left(\chi_q^\dagger\chi_q - 1/2\right) \qquad \varepsilon_q = \sqrt{(h/2-\cos q)^2 + \gamma^2\sin^2 q}$$

- Entanglement NOT of free fermions, but can be calculated using Toeplitz Determinants (Its et al. 2005)

Phase Diagram of the XY Model

$$H = -\sum_i \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

$$\varepsilon_q = \sqrt{(h/2 - \cos q)^2 + \gamma^2 \sin^2 q}$$

Phase Diagram:

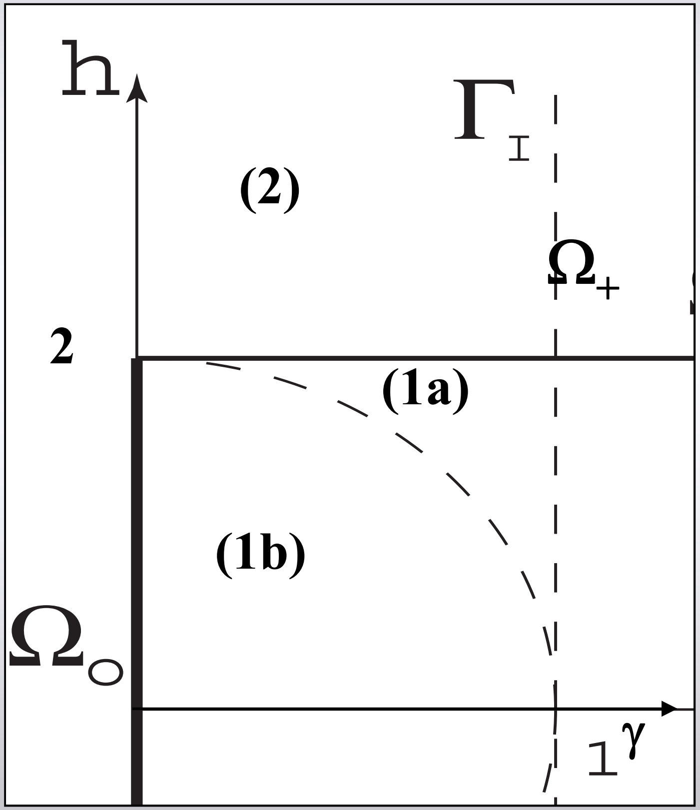
- 3 non-critical regions (**2,1a,1b**)
- 2 critical phases:
  - Ω_0: Isotropic XY
  - Ω_+: Critical magnetic field

For h < 2 (**1a,1b**) :

Spectrum Doubly Degenerate:

- **|GS>$_+$** : Even # Excitations
- **|GS>$_-$** : Odd # Excitations

We consider only **|GS>$_+$**

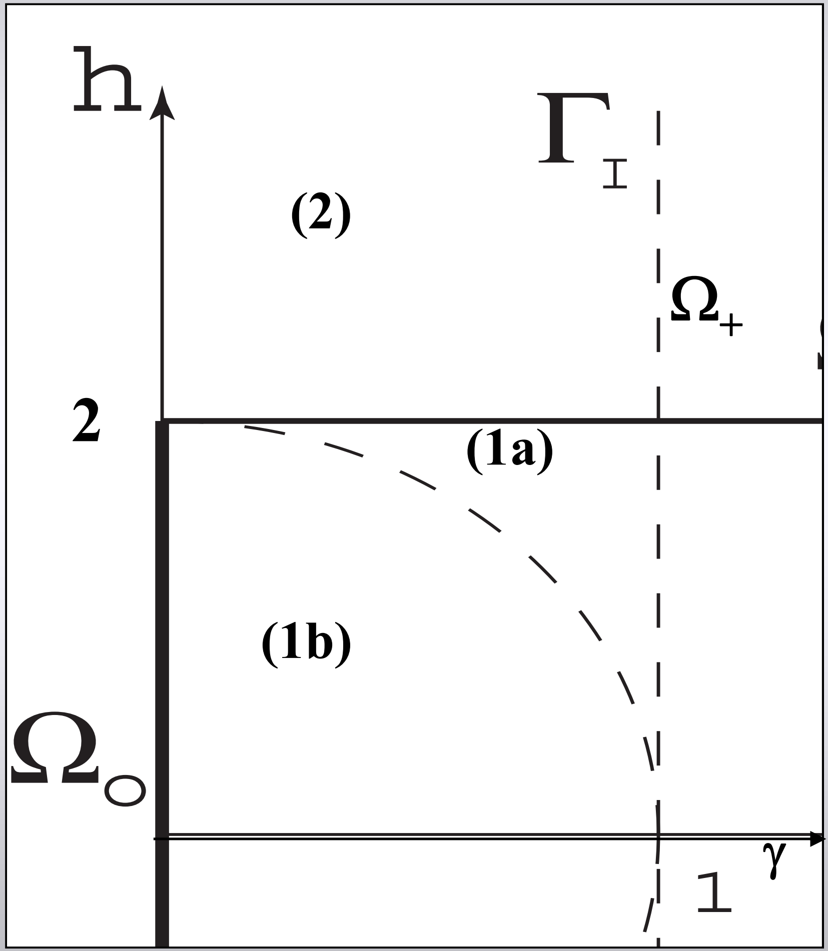


Phase Diagram of the XY Model
(only γ>0 shown)

Asymptotic Entropy of the XY model

- We have complete analytical expressions for the asymptotic entropy

- But let's discuss physics first (very interesting math, we'll discuss it later)



Minima of the Entropy

- Absolute minimum at $\mathbf{h} \to \infty$ or $\gamma \to 0$ $(\mathbf{h} > 2)$: $\mathbf{S}_\infty \to \mathbf{0}$

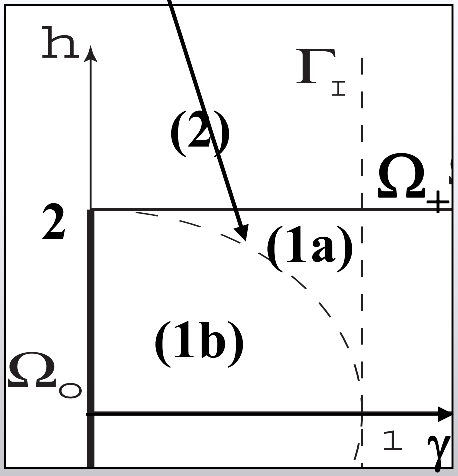
  as the ground state becomes ferromagnetic $(\uparrow \ldots \uparrow)$

- Local minimum $\mathbf{S}_\infty = \ln 2$
  at the boundary between cases **1a** and **1b** $(\gamma^2 + (h/2)^2 = 1)$

$\rightarrow$ Ground state almost factorized:

$$|GS_+\rangle = |GS_1\rangle + |GS_2\rangle$$

(each state is factorized and has <u>no entropy</u>)

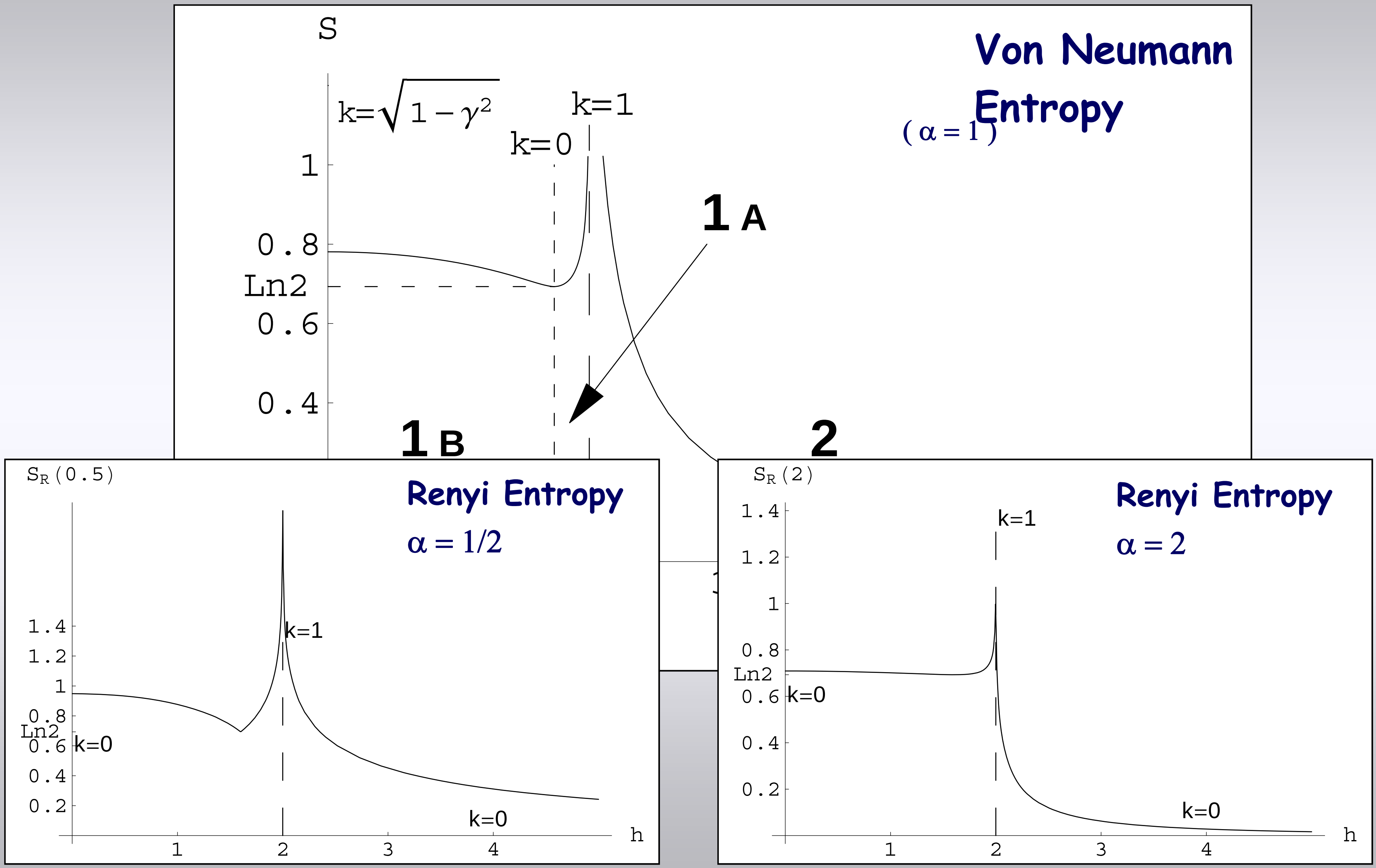


$$|GS_1\rangle = \prod_{n \in \mathrm{lattice}} [\cos(\theta)|\uparrow_n\rangle + \sin(\theta)|\downarrow_n\rangle]$$

$$|GS_2\rangle = \prod_{n \in \mathrm{lattice}} [\cos(\theta)|\uparrow_n\rangle - \sin(\theta)|\downarrow_n\rangle]$$

$$\cos^2(2\theta) = (1-\gamma)/(1+\gamma)$$

Entropy at fixed γ



Entropy on the critical phases

Phase transitions: as the gap closes $S_\infty \to +\infty$

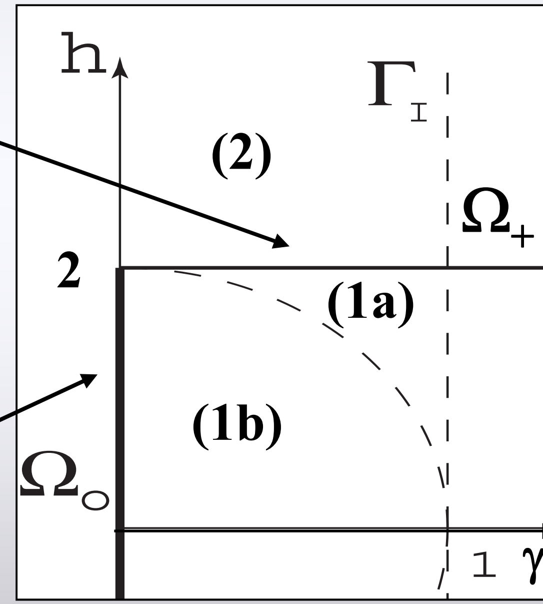
$$S_R(\alpha) = \frac{1+\alpha}{\alpha}\left(-\frac{1}{12}\ln|2-h| + \frac{1}{6}\ln 4\gamma\right) + O(|h-2|)$$

$$S_\infty \to -\frac{1}{6}\ln|2-h| + \frac{1}{3}\ln 4\gamma + O\left(|h-2|\ln^2|h-2|\right)$$

$h \to 2, \gamma \neq 0$

Critical Magnetic Field:

(Calabrese, Cardy 2004)



Isotropic XY model (XX Model):

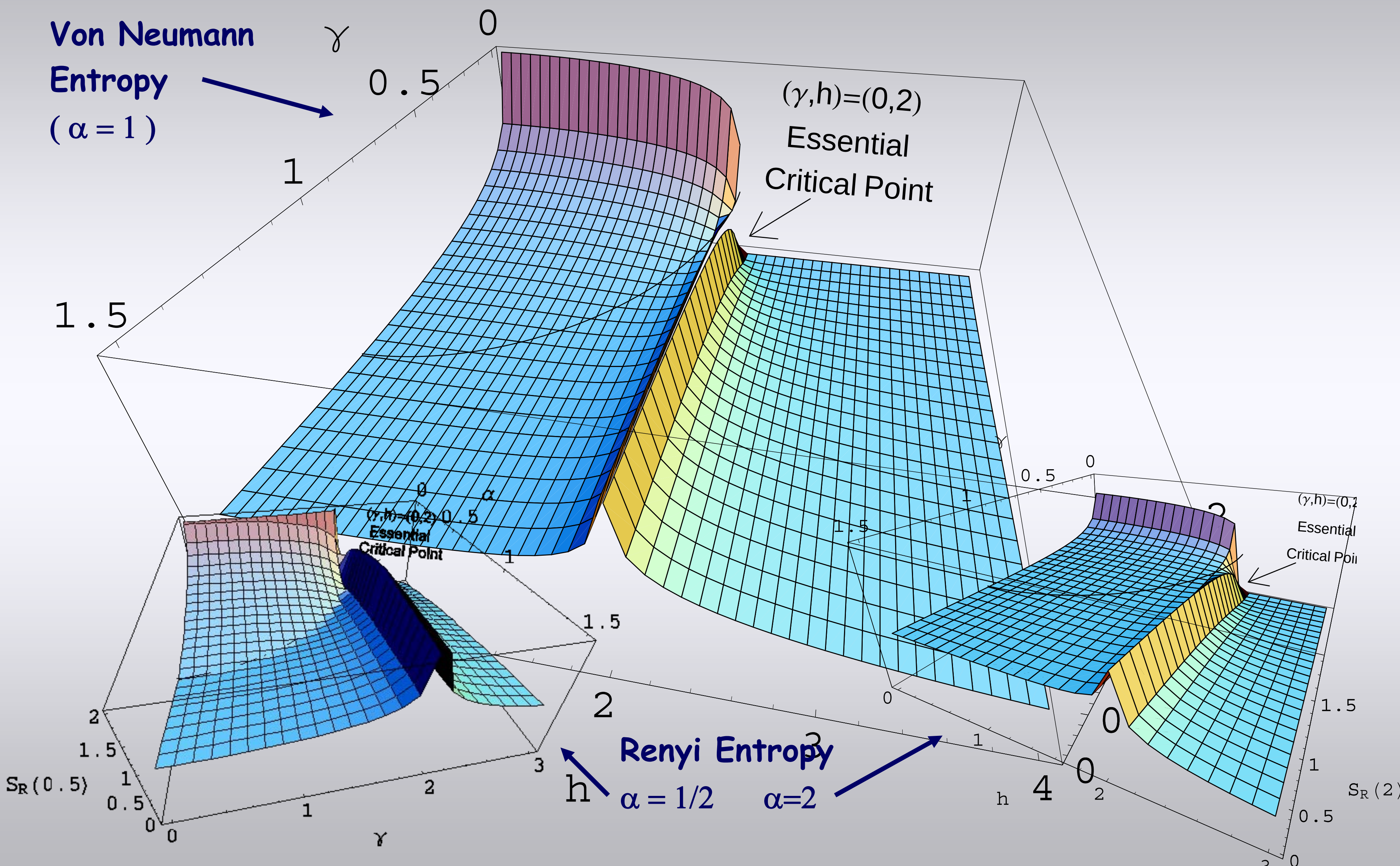
(Jin, Korepin 2003)

$\gamma \to 0, |h| < 2$

$$S_R(\alpha) = \frac{1+\alpha}{\alpha}\left(-\frac{1}{6}\ln\gamma + \frac{1}{12}\ln(4-h^2) + \frac{1}{6}\ln 2\right) + O(\gamma\ln^2\gamma)$$

$$S_\infty \to -\frac{1}{3}\ln\gamma + \frac{1}{6}\ln(4-h^2) + \frac{1}{3}\ln 2 + O(\gamma\ln^2\gamma)$$

3-D plot of the Entropy



The Essential Critical Point (ECP)

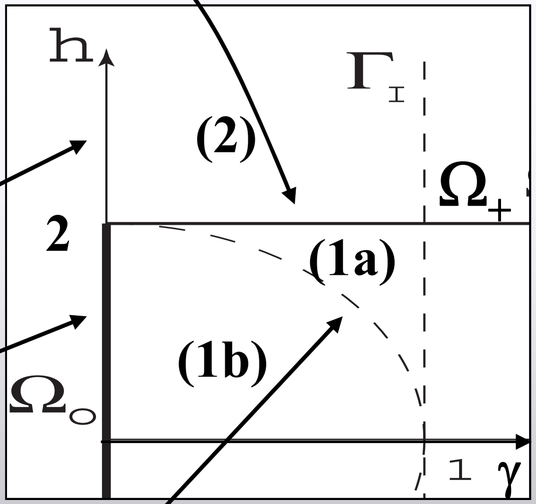
- Point $(\gamma, h) = (0,2)$ is a bi-critical point
- Gapless, but not conformal (quadratic spectrum)
- Let's study the entropy close to this point:

– Approaching it along $h=2,\ \gamma>0$: $S_\infty = \infty$

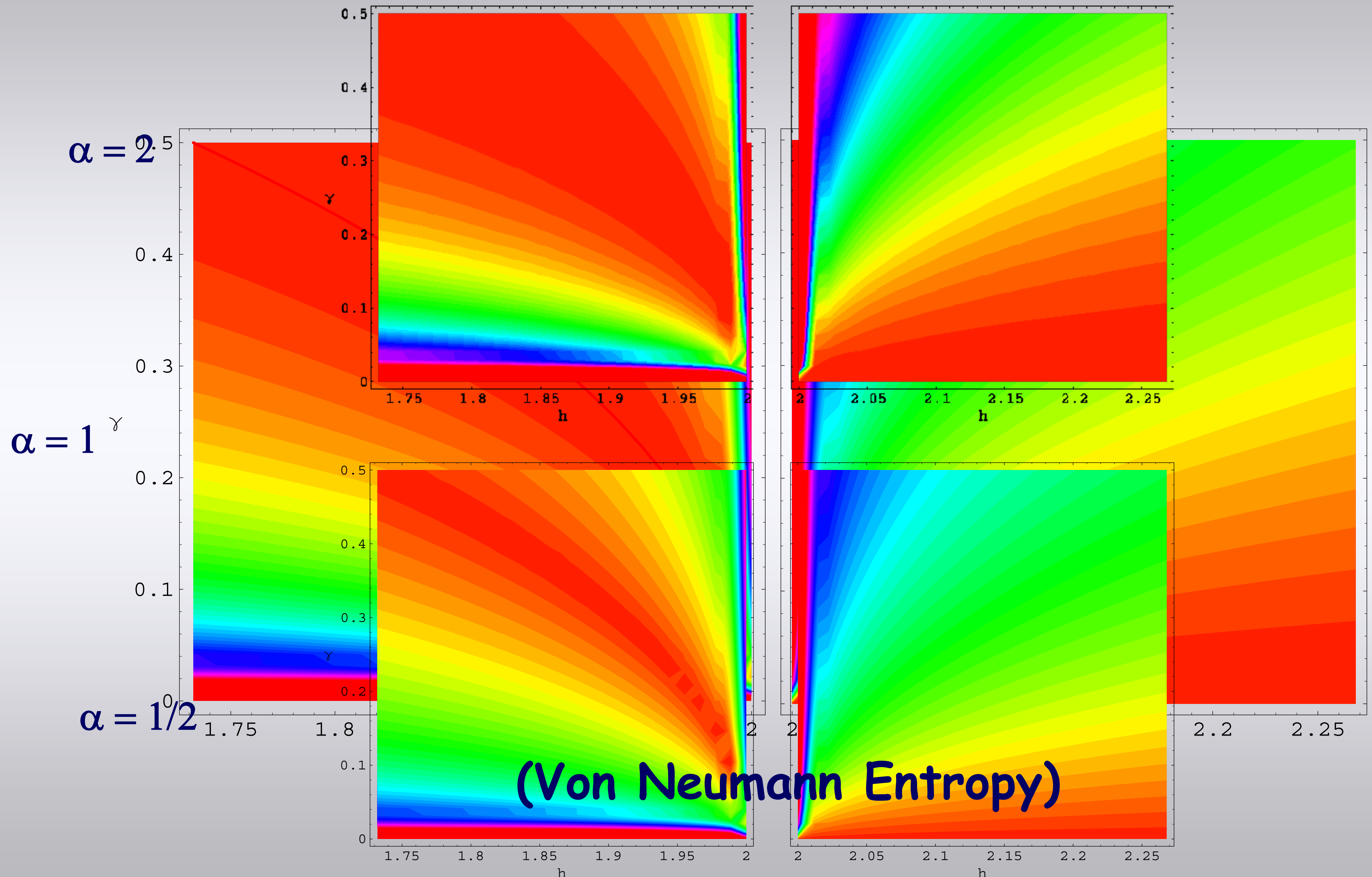
– Approaching it along $\gamma=0,\ h>2$: $S_\infty = 0$

– Approaching it along $\gamma=0,\ h<2$: $S_\infty = \infty$

– Approaching it along $\gamma^2 + (h/2)^2 = 1$: $S_\infty = \ln 2$

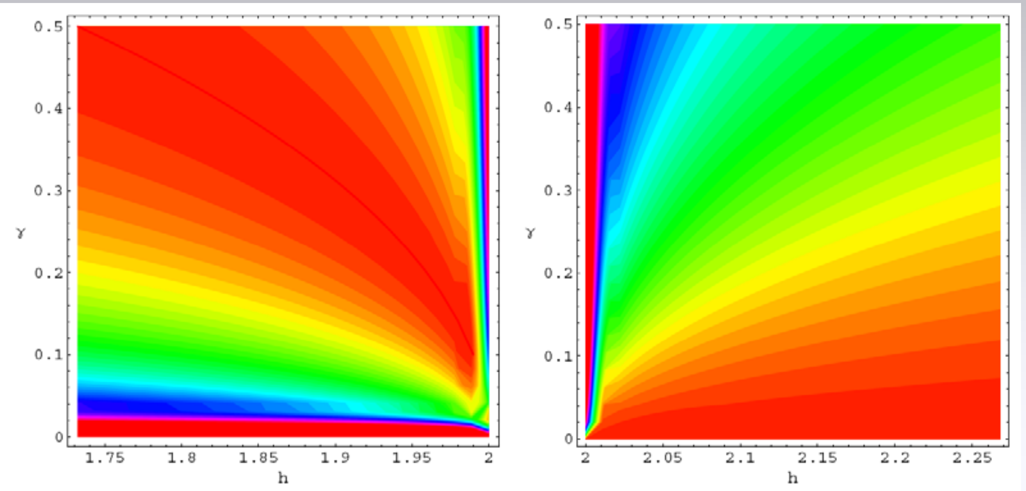


Entropy around the ECP

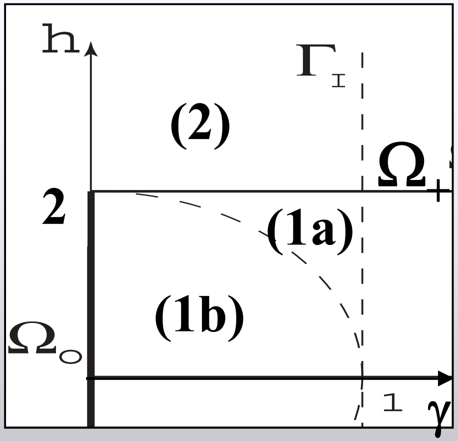


Curves of constant Entropy

- Curves of constant entropy are simple Hyperbolae and Ellipses:



Case 2 $(h > 2)$:	$\left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1,$		$0 \leq \kappa < \infty$
Case 1a $(2\sqrt{1-\gamma^2} < h < 2)$:	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1,$		$\kappa > 1$
Case 1b $(h < 2\sqrt{1-\gamma^2})$:	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1,$		$\kappa < 1.$

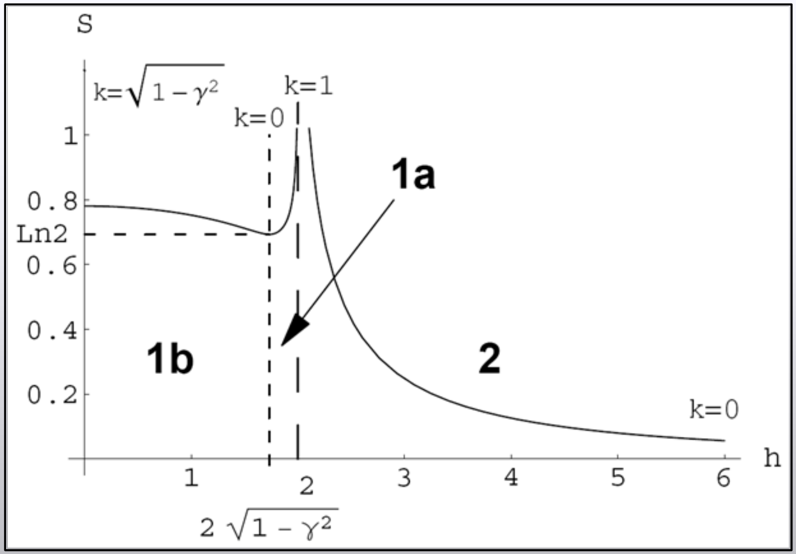


All these curves <u>pass through</u> the Essential Critical Point!

Importance of the Essential Critical Point

- From any point in the phase diagram one reaches the ECP following a curve of constant Entropy
- The range of the Entropy in the phase diagram is the positive real axis

$\Rightarrow$ Near the ECP the Entropy reaches every positive value!



- Small variations in the parameters change the Entropy dramatically
- ECP is an essential singularity for the entropy

Entropy vs. Correlation length

- XY model is simple: only 2 length scales

$$\lambda_\pm \equiv \frac{h \pm \sqrt{h^2+4\gamma^2-4}}{2(1+\gamma)}$$

- Spectrum:

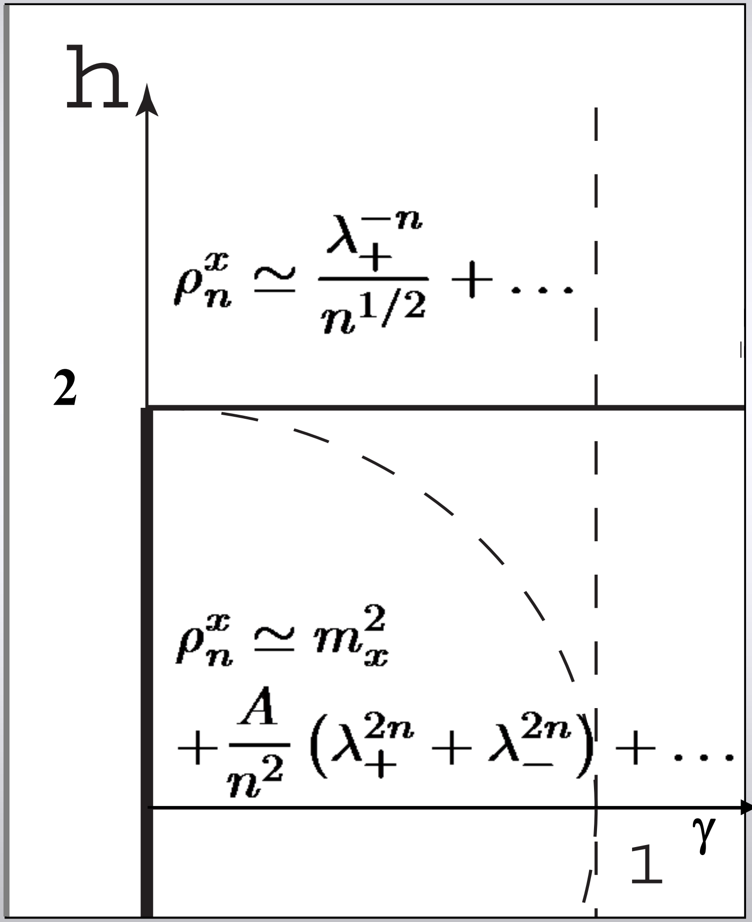
$$\begin{aligned}\varepsilon &= \sqrt{(h/2-\cos q)^2+\gamma^2\sin^2 q} \\ &= \sqrt{p(\mathrm{e}^{\mathrm{i}q})p(\mathrm{e}^{-\mathrm{i}q})}\end{aligned}$$

$$p(z) \equiv \frac{1+\gamma}{2z}(z-\lambda_+)(z-\lambda_-)$$

- 1st: Correlation length

  2nd: Subdominant;

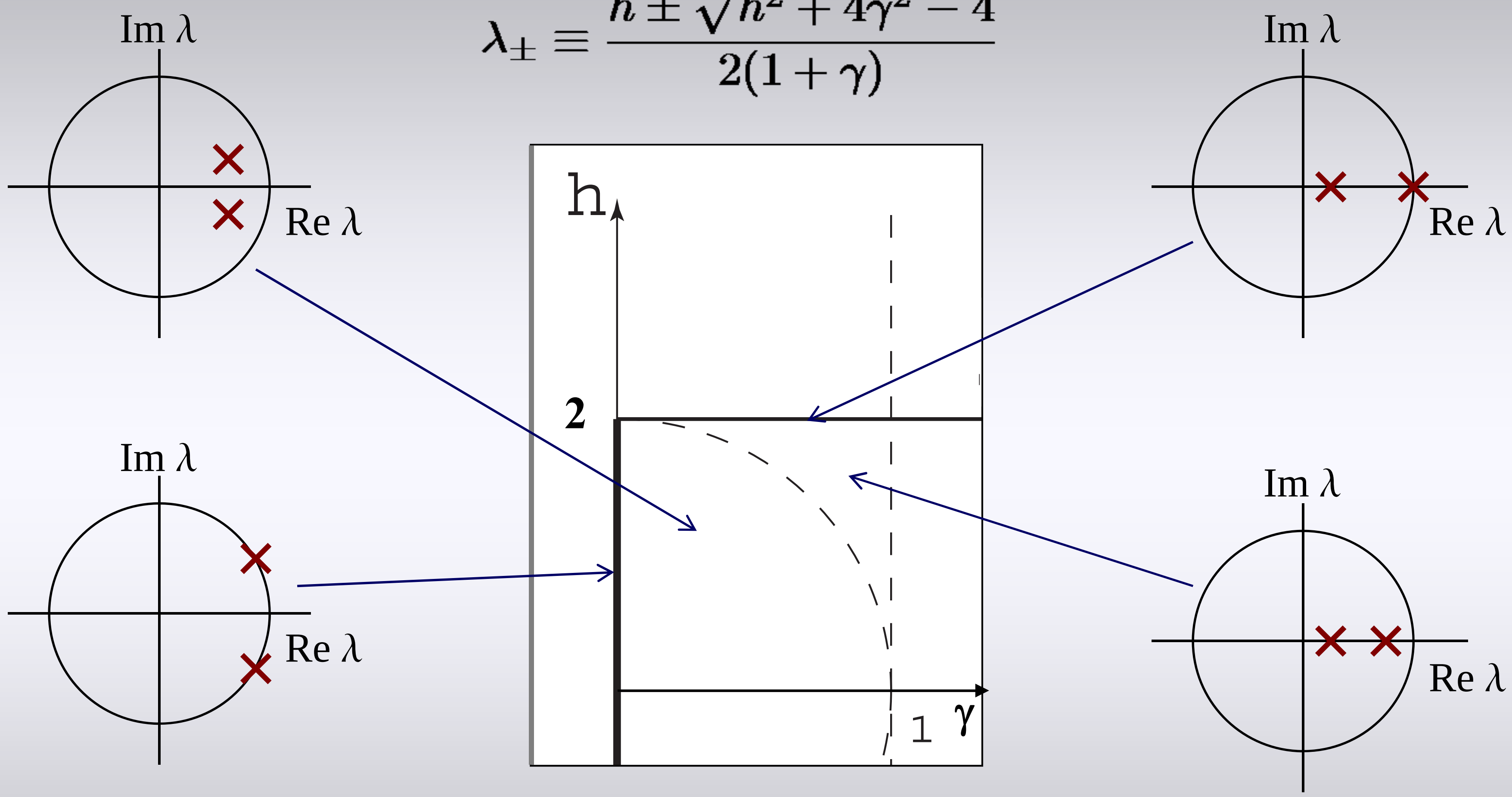
  Comm./Incummensurate



$$\rho_n^x = \langle \sigma_i^x \sigma_{i+n}^x \rangle$$

Entropy vs. Correlation length

$$\lambda_\pm \equiv \frac{h \pm \sqrt{h^2 + 4\gamma^2 - 4}}{2(1+\gamma)}$$



- Entropy depends on $k = k(\lambda_+, \lambda_-)$

  $\Rightarrow$ sensitive to both and their competition (ECP)!

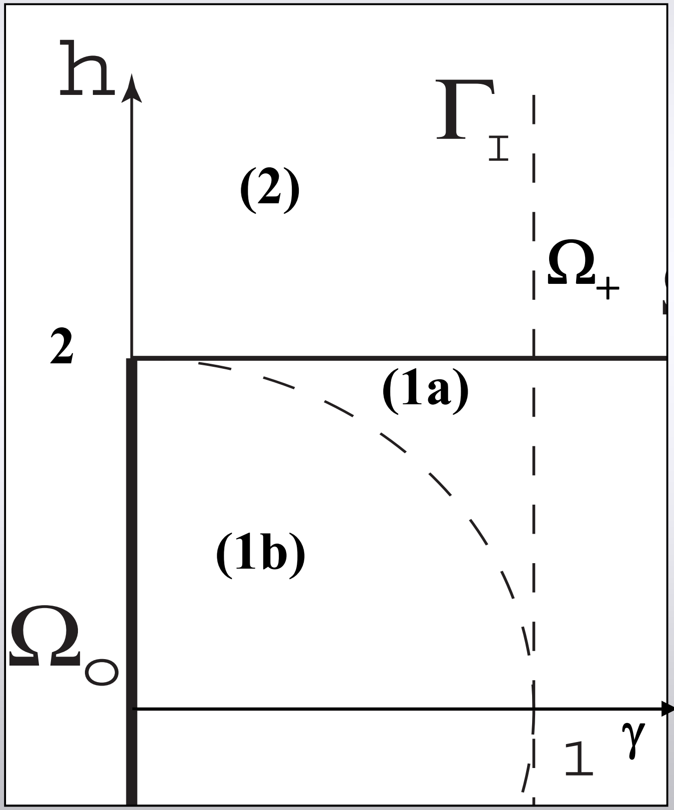
XY Model Recap

$$H = -\sum_i \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

- Saturates in gapped phases
- Correct logarithmic scaling close to conformal points
- Essential Singularity approaching non-conformal bi-critical point

  → due to 2 competing diverging length scales

  → order of limits: universality?



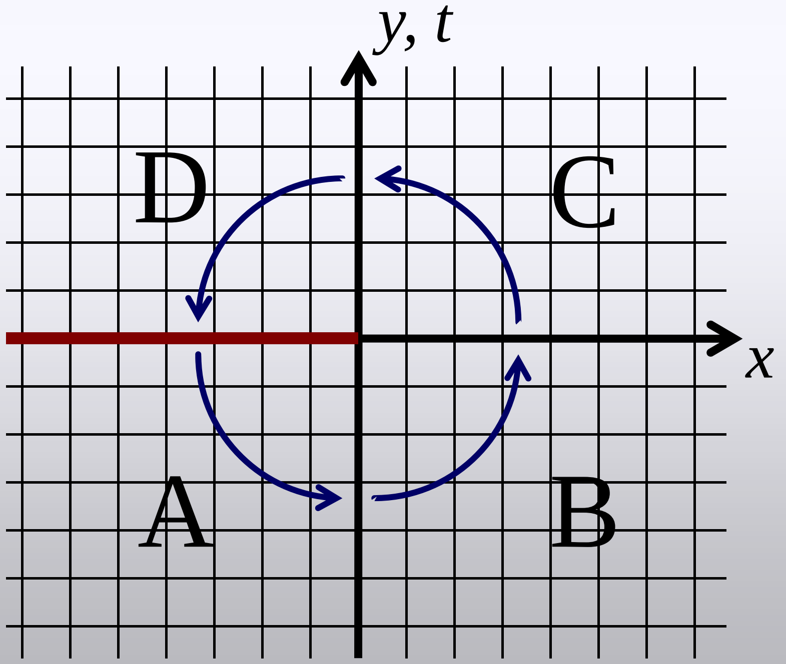
XYZ Spin Chain

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right)$$

- Commutes with transfer matrices of 8-vertex model
- Use of Baxter's Corner Transfer Matrices (CTM)

$$\mathcal{Z} = \mathrm{tr}\,(ABCD)$$

$$\rho_{\sigma,\sigma'} = (ABCD)_{\sigma,\sigma'}$$



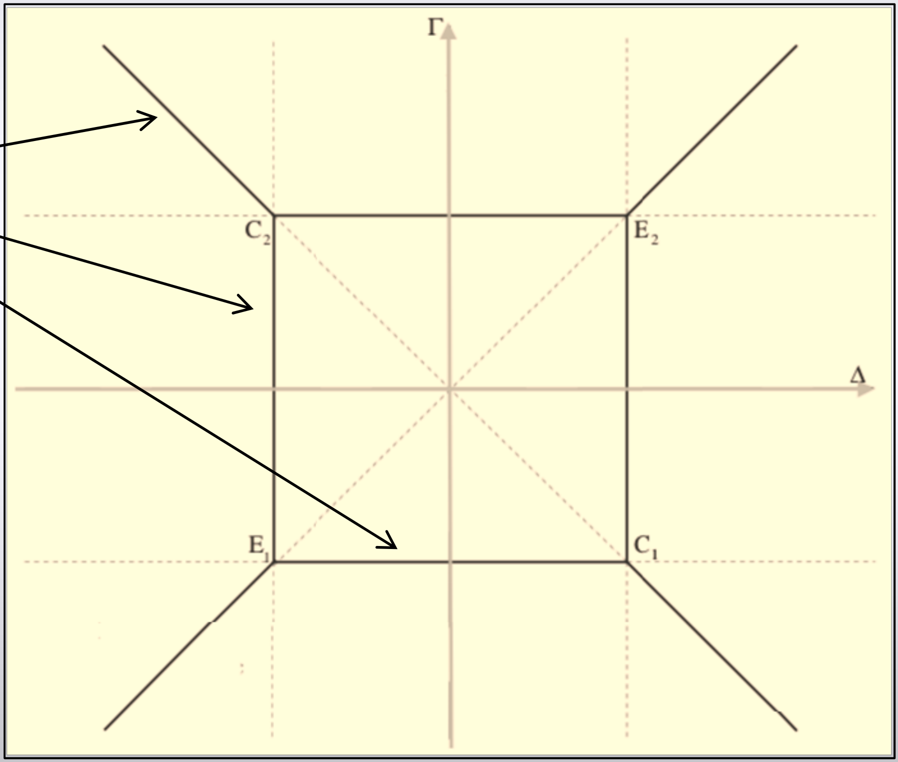
Phase Diagram of XYZ model

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right)$$

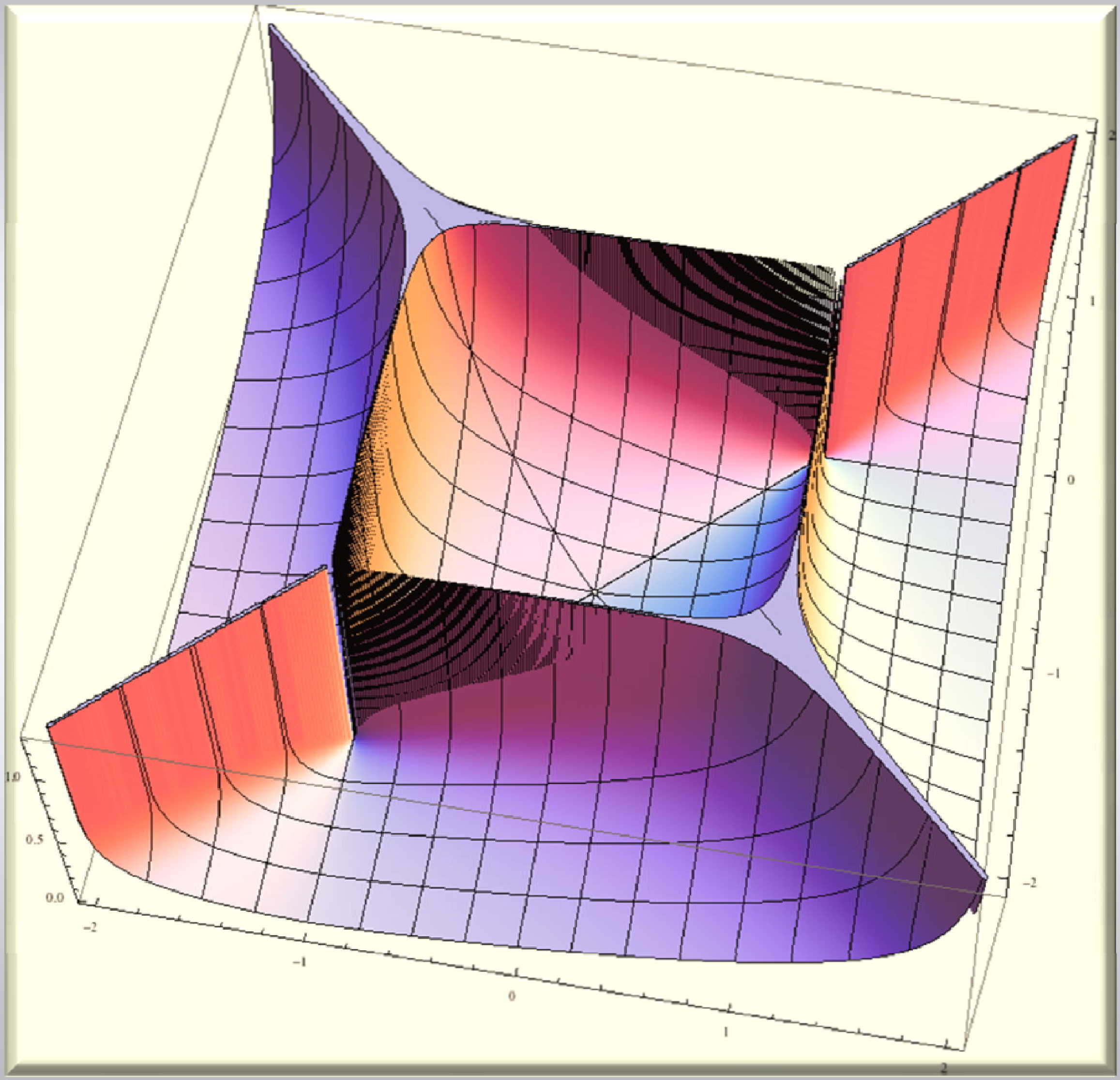
- Gapped in bulk of plane
- Critical on dark lines (rotated XXZ paramagnetic phases)
- 4 "tri-critical" points:

  $C_{1,2}$ conformal

  $E_{1,2}$ quadratic spectrum

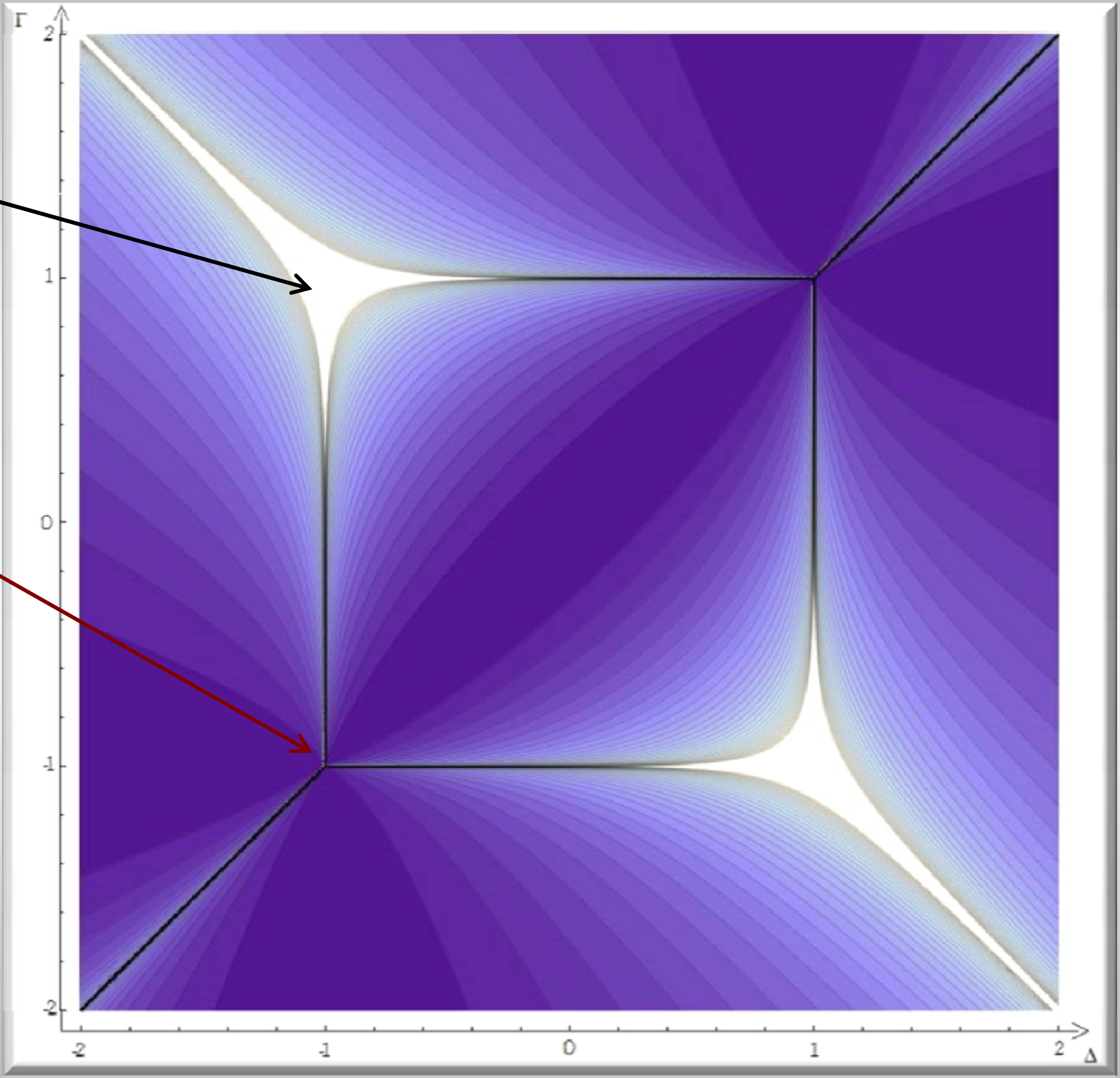


3-D plot of entropy



Iso-Entropy lines

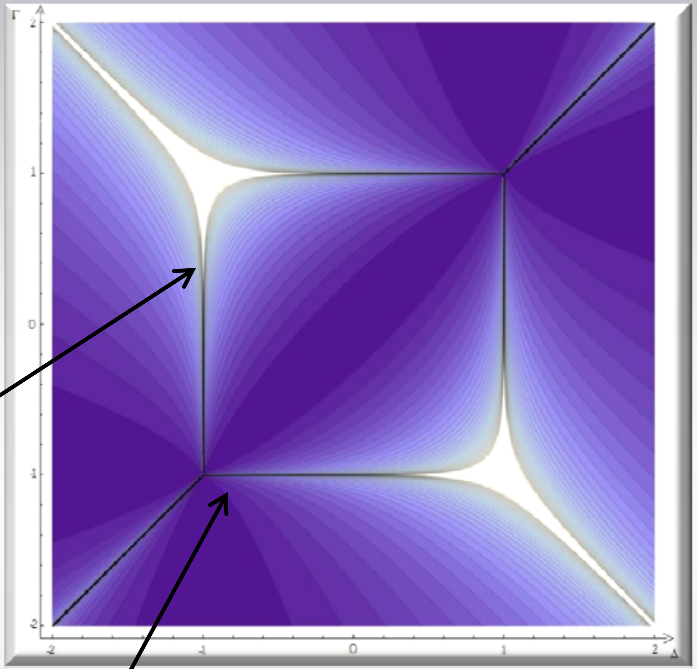
- Conformal point: entropy diverges close to it
- Non-conformal point (ECP): entropy goes from 0 to ∞ arbitrarily close to it (depending on direction)



Conformal check

- Expansion close to conformal points agree with expectations:

$$S_\alpha =$$
$$\frac{1}{12}\left(1+\frac{1}{\alpha}\right)\ln(\xi) - \frac{1}{24}\left(11-\frac{1}{\alpha}\right)\ln 2$$
$$+\frac{\alpha}{6(1-\alpha)}\left[\frac{\xi^{-2}}{4} + \mathrm{O}(\xi^{-4})\right]$$
$$-\frac{1}{6(1-\alpha)}\left[4(4\xi)^{-2/\alpha} + \mathrm{O}\left(\xi^{-4/\alpha}\right)\right]$$



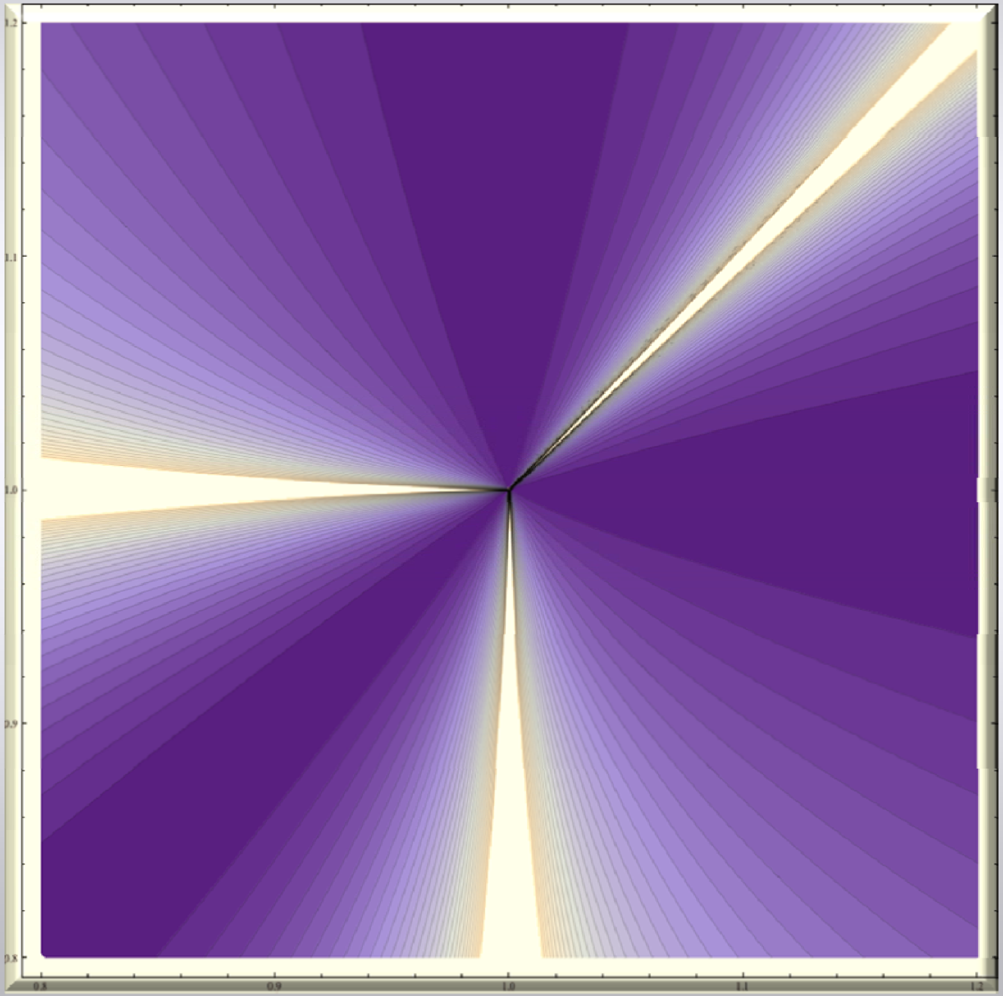
- Corrections constant while low-energy excitations are free spinons. Different corrections close to ferro point

$(\Delta > 0)$

Close-up to non-conformal point

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right)$$

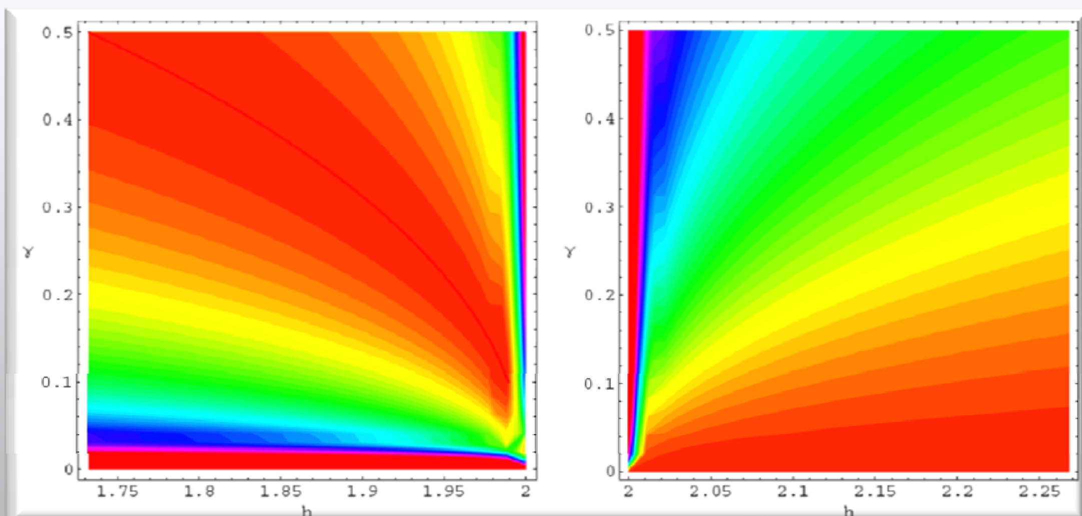
- Isotropic Ferromagnetic Heisenberg: quadratic spectrum
- Curves of constant entropy pass through it
- Similar physics as XY model

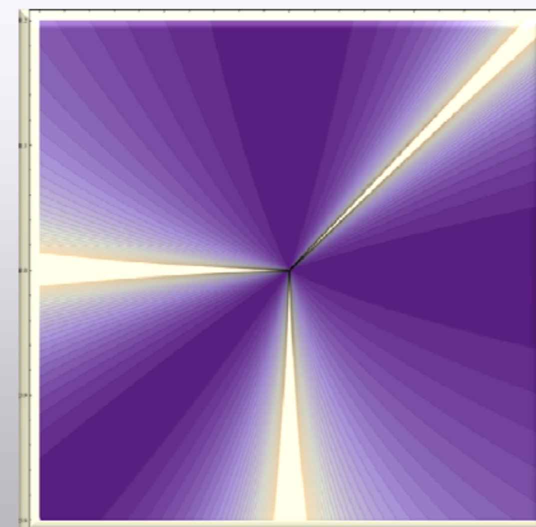


Physics round-up

- Gapped phases saturate
- Close to conformal points: logarithmic divergence
- Close to non-conformal points: <u>essential singularity</u> (entropy depends on direction of approach)

→ **Entanglement to discriminate non-conformal QPTs**
(finite size scaling?)





The technical side of the story

- Entanglement as interesting (new) correlator
- Potential tool to distinguish conformal and non-conf. QPT

also

- Non-local function, sensitive to global properties, but simpler than partition function
- Special structures arise from integrability:

regular entanglement spectrum

& modular symmetries

The Reduced Density Matrix

- All spin 1/2 integrable chain systems have the same diagonal structure for ρ (Baxter's Book, Peschel et al 2009, …):

$$\rho_d = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x_3 \end{pmatrix} \otimes \ldots$$

$$x_j = \begin{cases} \mathrm{e}^{-2j\epsilon}, & \text{ordered} \\ \mathrm{e}^{-(2j-1)\epsilon}, & \text{disordered} \end{cases}$$

where ϵ is characteristic of the model

Entanglement Spectrum

$$\rho_d = \bigotimes_{j=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & x_j \end{pmatrix} \qquad x_j = \begin{cases} \mathrm{e}^{-2j\epsilon}, & \text{ordered} \\ \mathrm{e}^{-(2j-1)\epsilon}, & \text{disordered} \end{cases}$$

- Eigenvalues form a geometric series
- Degeneracies from partitions of integers

  (Okunishi et al. 1999; Franchini et al. 2010; ...)

- All these models have the same entanglement spectrum

$$\rho = \mathrm{e}^{-\mathcal{H}_{\mathbf{Entanglement}}}$$

$\rightarrow \mathcal{H}_{\text{Entanglement}}$: free fermions with spectrum ϵ

Renyi Entropy

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha - 1} \sum_{j=1}^{\infty} \ln\left(1 + q^{2j}\right) + \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \ln\left(1 + q^{2j\alpha}\right)$$

$$q \equiv \mathrm{e}^{-\epsilon}$$

- Series can be summed into elliptic theta functions:

$$\begin{aligned} \mathcal{S}_\alpha(\epsilon) \quad &= \quad \frac{\alpha}{6(1-\alpha)} \ln \frac{\theta_4(0,q)\theta_3(0,q)}{\theta_2^2(0,q)} \\ &\quad + \frac{1}{6(1-\alpha)} \ln \frac{\theta_2^2(0,q^\alpha)}{\theta_3(0,q^\alpha)\theta_4(0,q^\alpha)} - \frac{\ln(2)}{3} \end{aligned}$$

- Analytical properties suitable for studying entropy as a function of ϵ and α (common to all integrable theories)
- ϵ dependence on microscopical parameter to study in the whole phase diagram

Modular Properties

$$\mathcal{S}_\alpha = \frac{\alpha}{6(1-\alpha)} \ln \frac{\theta_4(0,q)\theta_3(0,q)}{\theta_2^2(0,q)} + \frac{1}{6(1-\alpha)} \ln \frac{\theta_2^2(0,q^\alpha)}{\theta_3(0,q^\alpha)\theta_4(0,q^\alpha)} - \frac{\ln(2)}{3}$$

- Theta functions transform simply under Modular group

$$q = \mathrm{e}^{\mathrm{i}\pi\tau}, \qquad T:\ \tau \to \tau + 1, \qquad S:\ \tau \to -\frac{1}{\tau}$$

- Entanglement as modular function (for subgroup)
- Same (but more explicit) properties as Partition Function
- Modular transformation acts in phase-space
  (while in CFT they are real space)
- CTM – CFT connection? (Tokyo Group, Calabrese et al. 2010)

Characters

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha-1} \ln \prod_{j=1}^{\infty} \left(1+q^{2j}\right) + \frac{1}{1-\alpha} \ln \prod_{j=1}^{\infty} \left(1+q^{2j\alpha}\right)$$

- For integrable models: entropy reads characters
- CTM spectrum = Virasoro representation (Tokyo Group, Cardy...)
- Close to QPT: expansion in the T-dual variable (Cardy '10)
- In progress:

  role of $\xi^{-1} = f(q)$ in the subleading terms

Minimal Models (with A. De Luca)

- An RSOS Models:

$$S_\alpha = \frac{1}{1-\alpha} \ln \frac{Z_\alpha}{Z_1^\alpha} \qquad\qquad Z_\alpha = \sum_a E_a X_{a,r,r+1}(q)$$

- Gapless limit $\rightarrow$ Minimal Models of CFT

- Boundary consitions can select an individual character

(Tokyo Group; Feverati, Pearce '03)

$$X_{s,r,r+1}(q) = q^{\frac{c}{24} - h_{r,s}} \chi_{r,s}(q)$$

- In progress:

  analysis of entropy and
  role of correlation lenght

Conclusions & Outlook

- Analytical study of bipartite entanglement of the XY and XYZ model
- Entropy diverges at critical phases, saturates to constant in gapped
- Near conformal points, entropy diverges logarithmically
- Near (multi-critical) non-conformal points, entropy reaches every positive value : essential singularity (ECP)

  → Universality close to ECPs?

- Entanglement spectrum common to all integrable models

  → Elliptic and modular properties

- Finite size scaling?

Thank you!