Towards the NLIE formulation of the AdS/CFT spectral problem

Árpád Hegedűs
KFKI-RMKI, Budapest

Bologna, 12-16 Sept 2011

Based on:
J. Balog, Á. Hegedűs: arXiv:1104.4054
arXiv:1106.2100

Outline

- Motivation
- A concept for an NLIE formulation
- Different formulations of the mirror TBA
- Quasi-local formulation
- Conclusion

Motivation

- Desire: transform the TBA of AdS/CFT into a finite component NLIE

Kazakov et al. ’11

Balog,Hegedűs, work in progress

Why?

TBA has disadvantageous properties

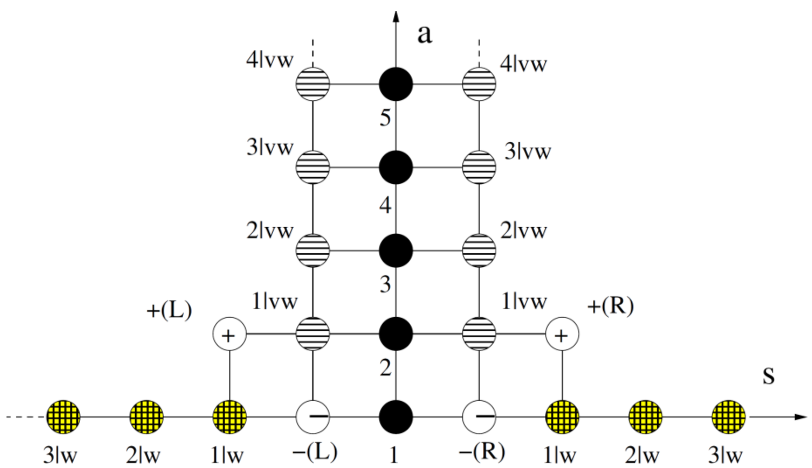
- Infinite components
- Due to discontinuities ⟶ non-local

„Non-local” infinitely many unknowns are joined by the eqs.

- Numerics: time consuming & hard to reach high precision

NLIE formulation:

- Higher numerical precision
- Less computational time



Concept of deriving an NLIE: the SU(2) case

- The SU(2) type cases: Klümper, Pearce '91, Destri de Vega '92, J. Suzuki '98 Dunning '03, Hegedűs '04, Balog, Hegedűs '09, R. Suzuki '11 etc. Gromov, Kazakov, Vieira '08

O(4) σ-model
O(3) σ-model
SG at $\beta \to 8\pi^2$

1-dimensional chain like TBA
1 massive node

TBA equations:

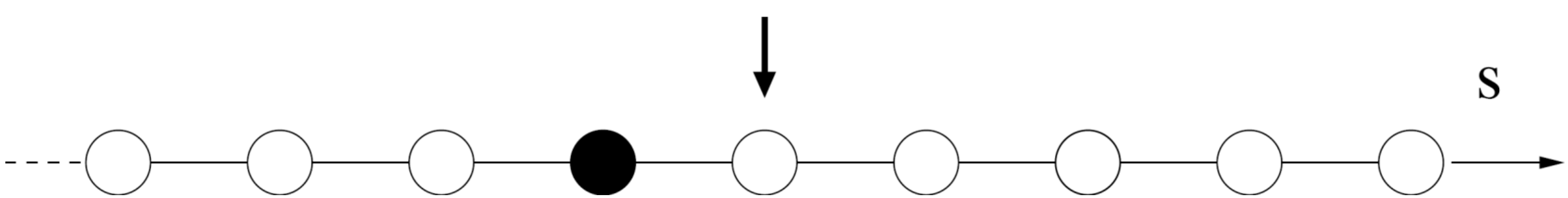
$$\ln Y_s = -\delta_{s,0}\, m\, L\, \cosh u + [\ln(1+Y_{s-1}) + \ln(1+Y_{s+1})] \star s \qquad s(u) \sim \frac{1}{\cosh u}$$

Y-system:

A. B. Zamolodchikov '91

$$Y_s^+ Y_s^- = (1+Y_{s-1})(1+Y_{s+1}) \qquad f^{\pm}(u) = f(u \pm i\frac{\pi}{2})$$

Diagramatic represenataion: O(4) σ-model

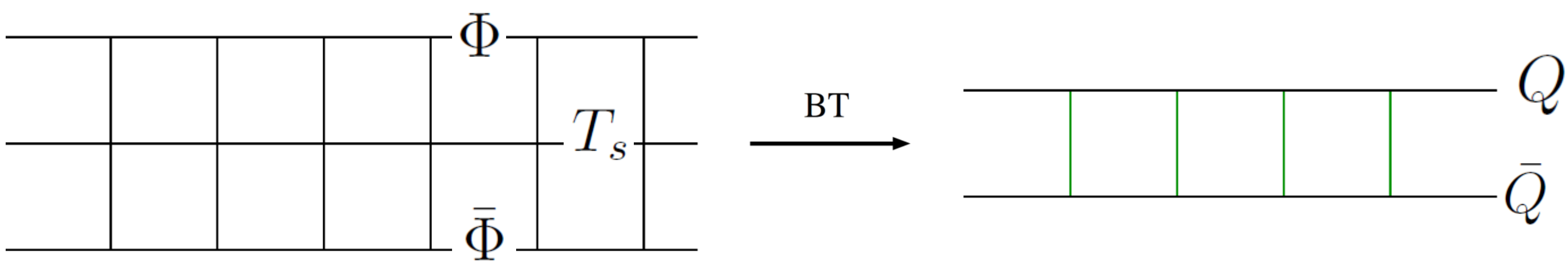


The SU(2) NLIE

T-system: discrete Hirota equations

$$T_s^+ \, T_s^- = \Phi^{[s]} \, \bar{\Phi}^{[-s]} + T_{s-1} \, T_{s+1} \qquad f^{[\pm k]}(u) = f(u \pm i\, k \, \frac{\pi}{2})$$

Solving T-system via Backlund transformations: Krichever, Lipan, Wiegmann, Zabrodin '96



TQ-relations:

$$T_{s+1} \, Q^{[s]} - T_s^- \, Q^{[s+2]} = \Phi^{[s]} \, \bar{Q}^{[-s-2]}$$

$$T_{s-1} \, \bar{Q}^{[-s-2]} - T_s^- \, \bar{Q}^{[-s]} = -\bar{\Phi}^{[-s]} \, Q^{[s]}$$

The SU(2) NLIE

All quantities are built from the elementary blocks:

TBA:

$$Y_s = \frac{T_{s-1}\,T_{s+1}}{\Phi^{[s]}\,\bar{\Phi}^{[-s]}} \qquad 1+Y_s = \frac{T_s^+\,T_s^-}{\Phi^{[s]}\,\bar{\Phi}^{[-s]}}$$

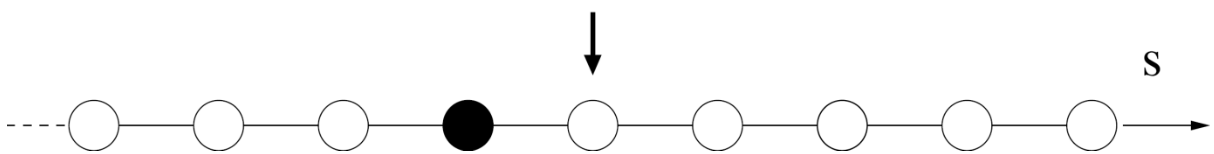
NLIE:

$$b_s = \frac{Q^{[s+2]}\,T_s^-}{\bar{Q}^{[-s-2]}\,\Phi^{[s]}} \qquad B_s = \frac{Q^{[s]}}{\bar{Q}^{[-s-2]}}\,\frac{T_{s+1}}{\Phi^{[s]}} \qquad B_s = 1 + b_s$$

Functional equations:

$$b_s\,\bar{b}_s = 1 + Y_s \qquad B_s^+\,\bar{B}_s^- = 1 + Y_{s+1}$$

TBA $\longrightarrow$ NLIE:



$$\ln Y_s = \ln(1+Y_{s-1}) \star s + \ln\underbrace{(1+Y_{s+1})}_{B_s^+\,\bar{B}_s^-} \star s$$

The SU(2) NLIE

Equations for b_s :

Definition of b_s

Analyticity properties of T_s , Φ , $\bar{\Phi}$, Q , $\bar{Q}$

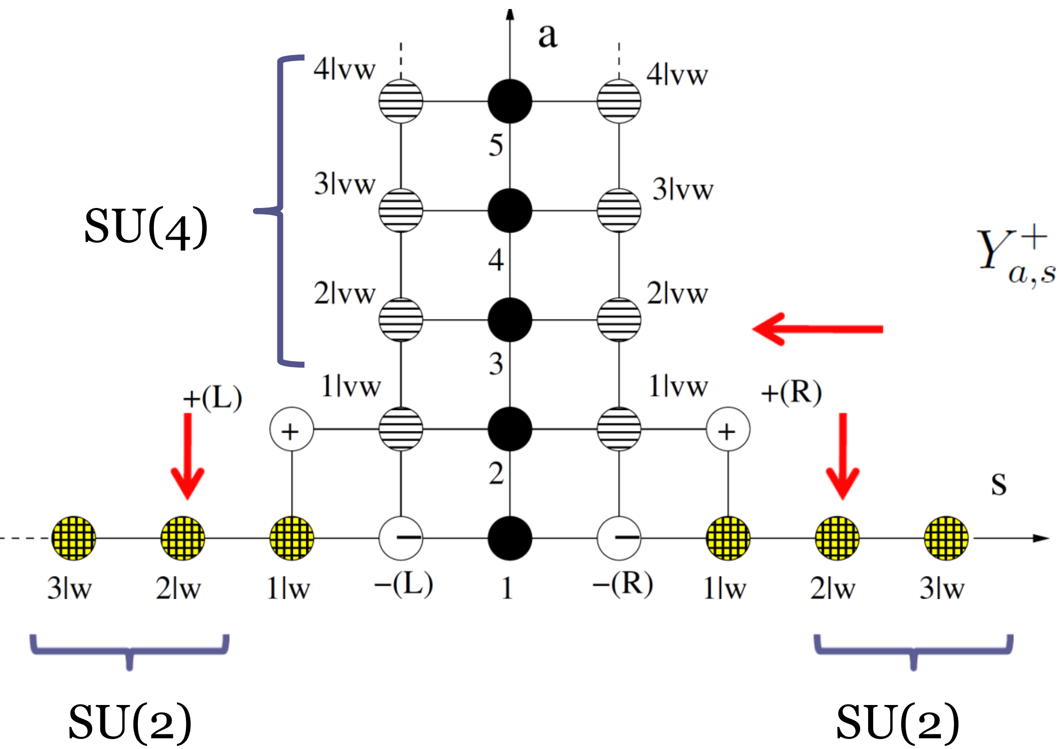
NLIE

$$\ln b_s = \ln B_s \star \ldots + \ln \bar{B}_s \star \ldots + \ln(1 + Y_s) \star \ldots$$

Important point:

Local formulation of the TBA equations!

The AdS/CFT case



Y-system: GKV '09

$$Y_{a,s}^{+}\,Y_{a,s}^{-}=\frac{(1+Y_{a,s-1})\,(1+Y_{a,s+1})}{(1+1/Y_{a-1,s})\,(1+1/Y_{a+1,s})}$$

$$Y_Q \sim e^{-L\mathcal{E}_Q}$$

Bombardelli, Fioravanti, Tateo '09
Gromov, Kazakov, Kozak, Vieira '09
Arutyunov, Frolov '09

TBA eqs: $$\ln Y_A = s_A + K_{AB}\star\ln(1+Y_B)$$

We need quasi-local TBA equations so that the SU(2) and SU(4) parts could be transformed into NLIE

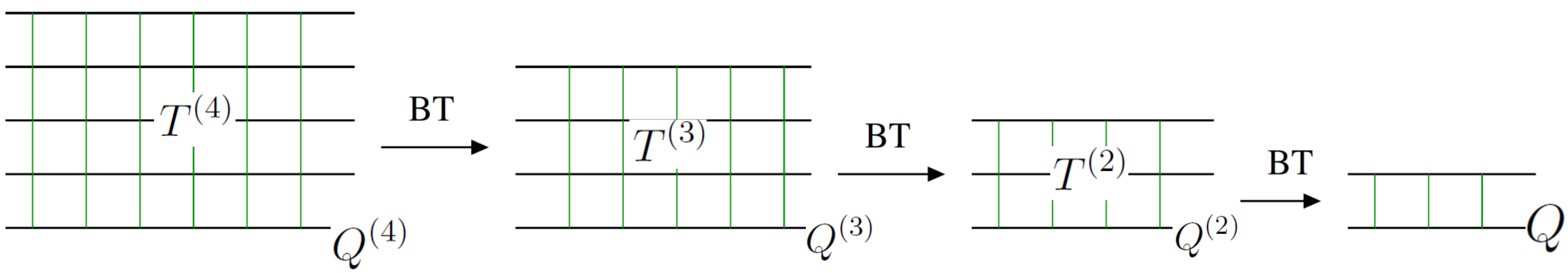
Quasi-locality: TBA eqs. generate only few neighbour interactions

SU(2) part: NLIE √ R. Suzuki '11

The SU(4) case

T-system: $T^+_{a,s}\,T^-_{a,s} = T_{a+1,s}\,T_{a-1,s} + T_{a,s+1}\,T_{a,s-1}$ $\qquad Y_{a,s} = \dfrac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$

Solution via Backlund transformations: Krichever, Lipan, Wiegmann, Zabrodin '96 + later others



Possible construction of an NLIE: Damareau, Klümper '06 Kazakov, Leurent '10

Build some $b^{(k)}_{a,s}$ functions from Ts and Qs

Functional relations: $''B_{a,s}\,\bar{B}_{a,s} = 1 + Y_{a+1,s}''$ $\qquad B_{a,s} = 1 + b_{a,s}$

Definition + analyticity information:

$$\ln b_{a,s} = \ln B_{a,s} \star \ldots + \ln(1 + Y_{a,s}) \star \ldots$$

The quasi-local formulation of the TBA for AdS/CFT

Extremalization of the mirror free energy

Canonical form (completely non-local)

Kernel identities

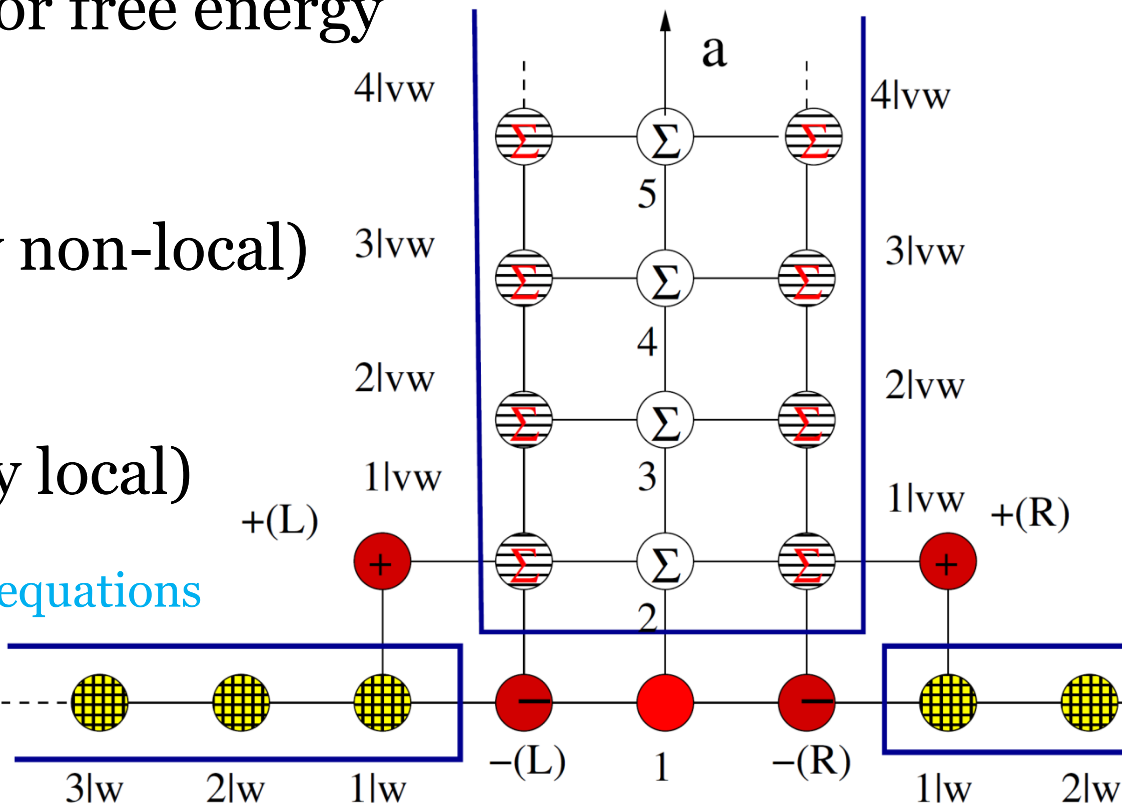
Simplified form (partially local)

Kernel identities + local equations

Hybrid form AFS '09

Kernel identities + local equations

Quasi-local form



From Canonical to Simplified TBA

Demonstration:

$$\ln Y^{(\alpha)}_{m|w} = \ln\left(1+\frac{1}{Y^{(\alpha)}_{m'|w}}\right)\star K_{m',m} - \ln\frac{1-\frac{1}{Y^{(\alpha)}_{+}}}{1-\frac{1}{Y^{(\alpha)}_{-}}}\star K_m \quad \Big/ \quad \sum_m \quad \ldots \quad \star\,(1+K)^{-1}_{m,n}$$

Kernel identities:

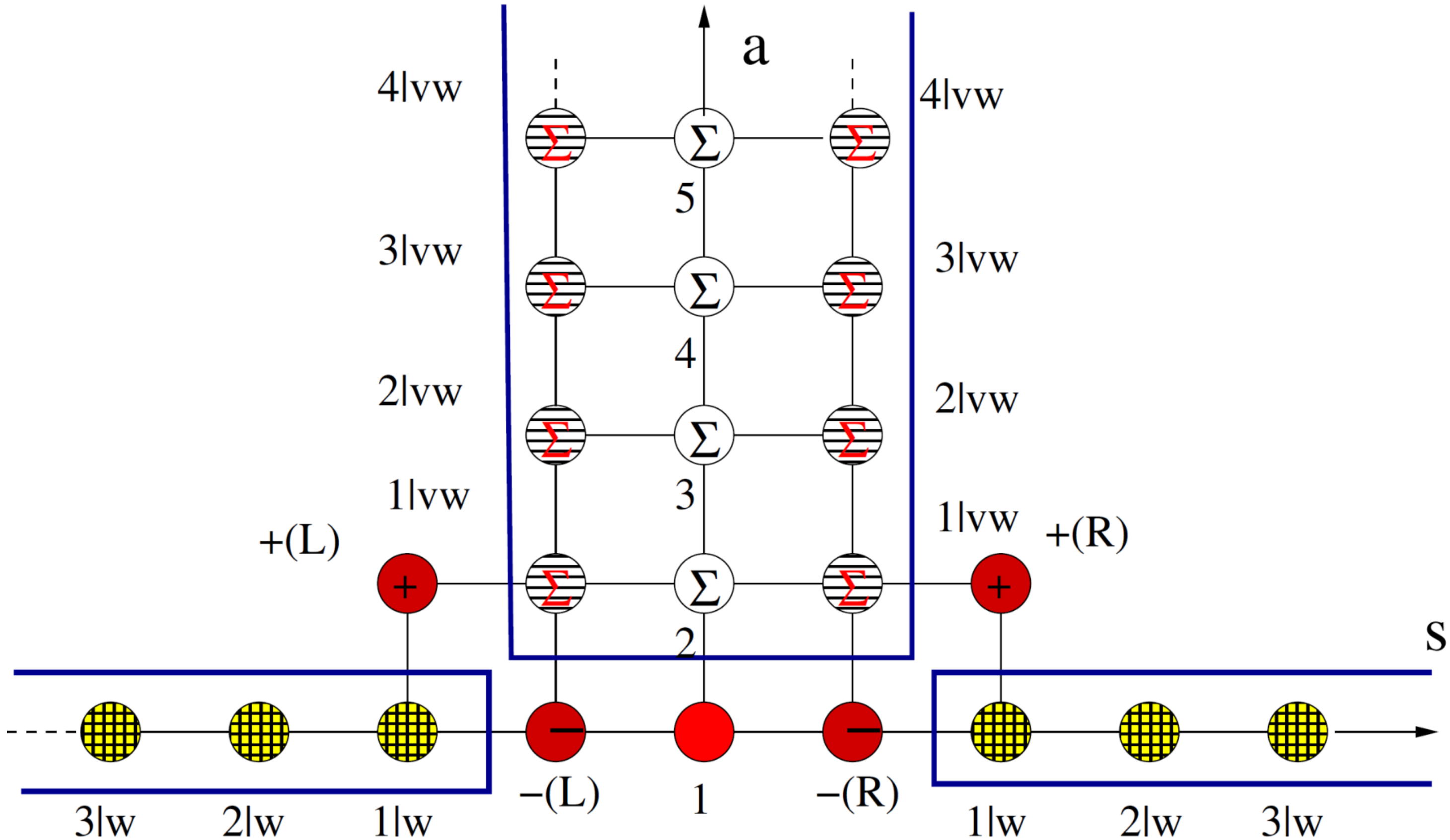
$$\sum_m (1+K)^{-1}_{n,m}\star K_m = s\cdot\delta_{n,1} \qquad (1+K)^{-1}_{n,m} = \delta_{n,m} - s\cdot I_{n,m}$$

$$I_{n,m} = \delta_{n-1,m} + \delta_{n+1,m} \qquad s(u) = \frac{g}{4}\frac{1}{\cosh\frac{\pi\, g\, u}{4}}$$

Simplified equation:

$$\ln Y^{(\alpha)}_{n|w} = \ln\left(1+Y^{(\alpha)}_{m|w}\right)\star s\cdot I_{m,n} - \ln\frac{1-\frac{1}{Y^{(\alpha)}_{+}}}{1-\frac{1}{Y^{(\alpha)}_{-}}}\star s\cdot\delta_{n,1}$$

Simplified TBA



$$\ln \frac{Y_+^{(\alpha)}}{Y_-^{(\alpha)}} = \sum_{Q=1}^{\infty} \ln(1+Y_Q) \star K_{Qy}$$

$$\ln Y_1 = -L\mathcal{E}_1 + \sum_Q \ln(1+Y_Q) \star \ldots + \sum_{m,\alpha} \ln\left(1+\frac{1}{Y_{m|vw}^{(\alpha)}}\right) \star \ldots + (Y_\pm^{(\alpha)} \quad \text{terms})$$

Simplified -> Hybrid TBA

Transformation of sums:

Arutyunov, Frolov, R. Suzuki '09

$$\sum_m \ln\left(1+\frac{1}{Y^{(\alpha)}_{m|vw}}\right)\star\ldots\to\sum_Q \ln(1+Y_Q)\star\ldots$$

Kernel identity:

$$\mathcal{K}_m - s\star(\mathcal{K}_{m-1}+\mathcal{K}_{m+1}) = \delta\mathcal{K}_m \qquad \delta\mathcal{K}_m\sim\delta_{m,\ldots}$$

Notations:

$$L_Q = \ln(1+Y_Q) \qquad L_{m|vw} = \ln\left(1+Y_{m|vw}\right)$$

$$\Lambda_Q = \ln\left(1+\frac{1}{Y_Q}\right) \qquad \Lambda_{m|vw} = \ln\left(1+\frac{1}{Y_{m|vw}}\right)$$

$$\ln Y_Q = L_Q - \Lambda_Q \qquad \ln Y_{m|vw} = L_{m|vw} - \Lambda_{m|vw}$$

Simplified -> Hybrid TBA

Local eqs.

$$\Lambda_{m|vw} = L_{m|vw} - (L_{m-1|vw} + L_{m+1|vw}) \star s + L_{m+1} \star s - f_2 \cdot \delta_{m,1}$$

$$f_2 = \ln\left(\frac{1-Y_-}{1-Y_+}\right) \hat{\star}\, s \qquad\qquad \Big/ \sum_m \ldots \star \mathcal{K}_m$$

Finally we get:

$$\sum_{m=1}^{\infty} \Lambda_{m|vw} \star \mathcal{K}_m = \underbrace{\sum_{m=1}^{\infty} L_{m|vw} \star \delta\mathcal{K}_m}_{\text{local}} + \sum_{m=1}^{\infty} L_{m+1} \star s \star \mathcal{K}_m - f_2 \star \mathcal{K}_1$$

Hybrid form:

$$\sum_m \Lambda_{m|vw} \star \ldots \to \sum_Q L_Q \star \ldots$$

Quasi-local formulation

Elimination of the terms: $$\sum_{Q} L_Q \star \ldots$$

Starting point: local equations:

$$L_Q = \Lambda_Q - s \star (\Lambda_{Q-1} + \Lambda_{Q+1}) + 2\Lambda_{Q-1|vw} \star s \qquad Q = 2, 3, \ldots$$

$$\Big/ \sum_{Q} \ldots \star \mathcal{K}_Q$$

We get:

$$\sum_{Q=2}^{\infty} L_Q \star \mathcal{K}_Q = 2\sum_{m=1}^{\infty} \Lambda_{m|vw} \star s \star \mathcal{K}_{m+1} + \underbrace{\sum_{Q=2}^{\infty} \Lambda_Q \star \delta\mathcal{K}_Q}_{\text{local}} + \Lambda_2 \star s \star \mathcal{K}_1 - \Lambda_1 \star s \star \mathcal{K}_2$$

Repeat the process:

$$\Lambda_{m|vw} = L_{m|vw} - (L_{m-1|vw} + L_{m+1|vw}) \star s + L_{m+1} \star s - f_2 \cdot \delta_{m,1}$$

$$\sum_{m=1}^{\infty} \Lambda_{m|vw} \star s \star \mathcal{K}_{m+1} = \underbrace{\sum_{m=1}^{\infty} L_{m|vw} \star s \star \delta\mathcal{K}_{m+1}}_{\text{local}} + \sum_{Q=2}^{\infty} L_Q \star s \star s \star \mathcal{K}_Q + L_{1|vw} \star s \star s \star \mathcal{K}_1 - f_2 \star s \star s \star \mathcal{K}_2$$

Quasi-local formulation

After elimination:

$$\sum_{Q=2}^{\infty} L_Q \star \mathcal{K}_Q = \sum_{Q=2}^{\infty} L_Q \star s \star s \star \mathcal{K}_Q + (\text{local terms})$$

Linear integral equation for $\sum_{Q=2}^{\infty} L_Q \star \mathcal{K}_Q$?

$s \star s$ to the right? $\quad s \star \mathcal{K}_Q \neq \mathcal{K}_Q \star s$

$s \star s$ to the left? $\quad \int_{-\infty}^{\infty} L_Q(u)\mathcal{K}_Q(u,v) = L_Q \star \mathcal{K}_Q(v)$

Possible solution:

$$L_Q(u) \to L_Q(u,v) \equiv L_Q(u-v)$$

Extend convolution:

$$(F_1 \circledast F_2)(u,v) = \int_{-\infty}^{\infty} dw\, F_1(u,w)\, F_2(w,v)$$

Quasi-local formulation

Property: if $F_1(u,v) = F_1(u-v)$ and $F_2(u,v) = F_2(u-v)$ then:

$$(F_1 \circledast F_2)(u,v) = (F_1 \star F_2)(u-v) = (F_2 \star F_1)(u-v) = (F_2 \circledast F_1)(u,v)$$

The modification of the result:

$$\sum_{Q=2}^{\infty} L_Q \circledast \mathcal{K}_Q = \sum_{Q=2}^{\infty} L_Q \circledast s \circledast s \circledast \mathcal{K}_Q + (\text{local terms})$$

$s \circledast s$ can be lifted to the left:

$$\sum_{Q=2}^{\infty} L_Q \circledast \mathcal{K}_Q = s \circledast s \circledast \sum_{Q=2}^{\infty} L_Q \circledast \mathcal{K}_Q + (\text{local terms})$$

Linear integral equation for $\Omega(u,v) = \sum_{Q=2}^{\infty} (L_Q \circledast \mathcal{K}_Q)(u,v)$

Explicitly solvable in Fourier space!

Quasi-local formulation

So we determined:
$$\Omega(u,v)=\sum_{Q=2}^{\infty}\int_{-\infty}^{\infty} L_Q(u-u')\,\mathcal{K}_Q(u',v)$$

We need:
$$\sum_{Q=2}^{\infty}\int_{-\infty}^{\infty} L_Q(u')\,\mathcal{K}_Q(u',v) \neq \Omega(0,v)=\sum_{Q=2}^{\infty}\int_{-\infty}^{\infty} L_Q(-u')\,\mathcal{K}_Q(u',v)$$

Correction: start from local TBA of $Y_{m|vw}(-u)$ and $Y_Q(-u)$

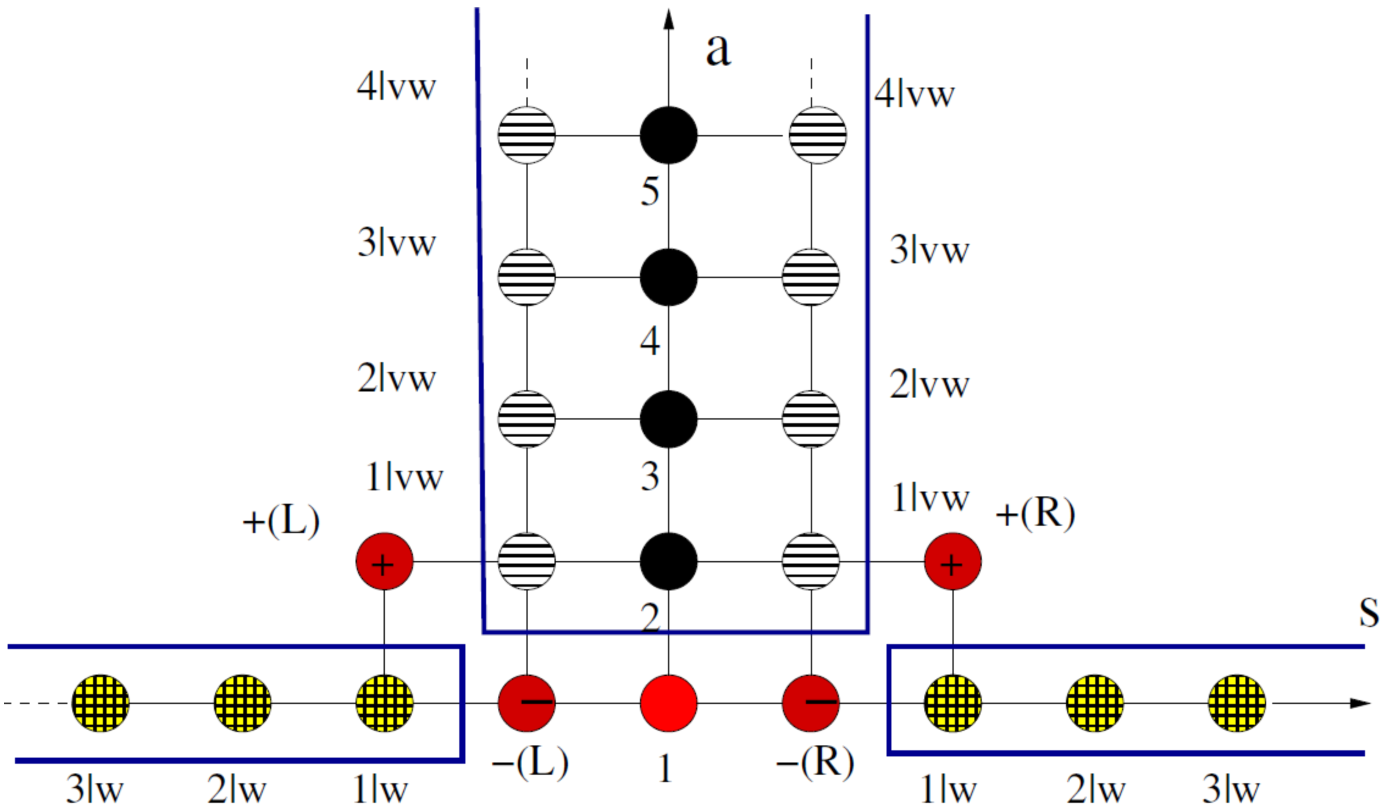
So can determine:
$$\hat{\Omega}(u,v)=\sum_{Q=2}^{\infty}\int_{-\infty}^{\infty} \hat{L}_Q(u-u')\,\mathcal{K}_Q(u',v) \quad \text{with} \quad \hat{L}_Q(u)=L_Q(-u)$$

$$\boxed{\hat{\Omega}(0,v)=\sum_{Q=2}^{\infty}\int_{-\infty}^{\infty} L_Q(u')\,\mathcal{K}_Q(u',v)}$$

Quasi-local formulation

Final result:

$$\sum_{Q=2}^{\infty} L_Q \star \mathcal{K}_Q = \text{LinearFunc}\left\{ \Lambda_1, \Lambda_2, \ln\left(\frac{1-Y_-^{(\alpha)}}{1-Y_+^{(\alpha)}}\right), L_{1|vw}^{(\alpha)} \right\}$$



Conclusions and work in progress

- We liberated the TBA eqs. of AdS/CFT from all highly non-local expressions and formulated them in a quasi-local form

- Advantages of this new formulation:
- Good starting point for an NLIE formulation
- Simplifies TBA equations & expressions for the physical quantities like the energy.

- Work in progress:
- Understanding analyticity properties of T's at all levels of nesting
- Creating appropriate NLIE variables
- Set up an NLIE