Ferminoic Basis of Local Operators in the sine-Gordon Model

Michio Jimbo (Rikkyo University, Japan)

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- JMS, J. Phys. A, **42** 304018 (2009)
- BJMS, Commun.Math.Phys. **299** 825–866 (2010)
- JMS, Lett.Math.Phys. **96** 325–365 (2011)
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TWO-POINT CORRELATION FUNCTION IN SCALING LEE–YANG MODEL

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The structure of the UV singularity in the two-point correlation function is considered for the scaling Lee–Yang model. Both perturbative and nonperturbative corrections to UV conformal theory are discussed. The IR convergent perturbation theory for the structure functions of operator algebra is developed and the first-order corrections are calculated explicitly. The UV expansion is compared numerically with the results of partial summation over intermediate asymptotic states. These two expansions match well in the intermediate region and give a reasonable precision data for the correlation function in the whole region of scaling distances.

1. Introduction

The conformal field theory (CFT) [1–3], being the theory of renormalization group fixed point, provides us (among other applications) with the classification of possible ultraviolet (UV) behavior in general relativistic field theory (RFT). Because of extremely high symmetry, CFT models are typically exactly solvable and today we have an enormous number of explicit constructions (see e.g. refs. [1–11]). From this point of view in approach to general RFT it is natural to begin with the short-distance CFT and consider the corresponding renormalization group trajectory as a perturbation of CFT model by a suitable relevant (or marginal) scalar operator (see e.g. refs. [12–20], where this approach was applied to different 2D problems). As a starting point one usually takes the conventional action

$$A_{\mathrm{RFT}} = A_{\mathrm{CFT}} + g\int \varphi(x)\,\mathrm{d}^2x\,, \tag{1.1}$$

where the scalar CFT field $\varphi(x)$ has dimension $\Delta(=\bar{\Delta}) \leqslant 1$. The coupling constant g develops positive scale dimension $g \sim (\text{mass})^{2-2\Delta}$ for $\Delta < 1$ and becomes dimensionless in the marginal case $\Delta = 1$. The last case must correspond to asymptotically free renormalization group behavior to make sense in the picture under consideration.

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Exracts from Al.Zamolodchikov

"Consider, for example, the perturbative expansion of a particular relativistic field theory (RFT) correlation function, say the two-point one,

$$\langle \Phi(x)\Phi(0)\rangle_{RFT},$$

where Φ is some local field. For simplicity it is supposed to be scalar. One may try the following formal expansion:

$$\langle \Phi(x)\Phi(0)\rangle_{RFT} = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int \langle \tilde{\Phi}(x)\tilde{\Phi}(0)\varphi(y_1)\cdots\varphi(y_n)\rangle_{CFT} d^2y_1\cdots d^2y_n,$$

where the correlation functions in the right-hand side are calculated in CFT and $\tilde{\Phi}(x)$ is the corresponding UV limit of the field $\Phi(x)$."

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"As for the IR divergencies, they cannot be absorbed into any local entities of the theory and lead to known non-analyticity in the coupling constant. In this paper we try to handle these nonperturbative corrections. The point of view is the following. To estimate UV behavior of e.g. the two point function, we start with the local operator product expansion

$$\Phi(x)\Phi(0) = \sum_i C^i_{\Phi\,\Phi}(x) A_i(0)\,,$$

where $A_i(0)$, $i = 1, 2, \cdots$ is the complete set of local fields in the theory and $C^i_{\Phi\,\Phi}(x)$ are the corresponding structure functions."

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It is important to determine the VEVs which encode all non-perturbative information.

Consider the sine-Gordon model

$$\mathcal{L}_{\mathrm{s}G} = \frac{1}{16\pi}(\partial_\mu\varphi)^2 - \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2}(e^{-i\beta\varphi} + e^{i\beta\varphi})$$

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We view this model as a perturbation of the complex Liouville model.
The primary fields are the exponential fields

$$\Phi_\alpha = e^{\frac{\nu\alpha}{2\sqrt{1-\nu}} i\varphi}\,.$$

Here and after we change the parameter β to

$$\nu = 1 - \beta^2 \quad (\frac{1}{2} < \nu < 1)\,.$$

Known results about VEV

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Primary field (LZ 1997)

$$\langle \Phi_\alpha(0)\rangle = [\Gamma(\nu)\boldsymbol{\mu}]^{\frac{\nu\alpha}{2(1-\nu)}}$$

$$\times \exp\Bigl(\int_0^\infty \Bigl(\frac{\sinh^2(\nu\alpha t)}{2\sinh(1-\nu)t\sinh t\cosh\nu t} - \frac{\nu^2\alpha^2}{2(1-\nu)}e^{-2t}\Bigr)\frac{dt}{t}\Bigr)\,.$$

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First non-trivial descendant (FFLZZ 1998)

$$\frac{\langle \ell_{-2}\bar{\ell}_{-2}\Phi_\alpha(0)\rangle}{\langle \Phi_\alpha(0)\rangle} = -\frac{(\Gamma(\nu)\boldsymbol{\mu})^{4/\nu}}{(1-\nu)^2} \frac{\gamma(-\frac{1}{2}+\frac{\alpha}{2}+\frac{1}{2\nu})}{\gamma(\frac{1}{2}+\frac{\alpha}{2}-\frac{1}{2\nu})} \frac{\gamma(\frac{\alpha}{2}-\frac{1}{2\nu})}{\gamma(\frac{\alpha}{2}+\frac{1}{2\nu})}$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} .$$

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where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,.$$

One would like to extend these calculations systematically.

For generic ν and α, the "Virasoro descendants"

$$\ell_{-\mathbf{N}}\bar{\ell}_{-\bar{\mathbf{N}}}\Phi_{\alpha}(0)$$

remain good as a label for the corresponding renormalized fields. Hoewever this is not the best basis for various reasons.

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We propose an alternative, fermionic basis of the same space

$$\mathbf{i}_K\bar{\mathbf{i}}_{\bar{K}}\boldsymbol{\beta}^*_{J^+}\boldsymbol{\gamma}^*_{J^-}\bar{\boldsymbol{\beta}}^*_{\bar{J}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{J}^-}\Phi_\alpha$$

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where $\sharp J^+ = \sharp J^-$, $\sharp \bar{J}^+ = \sharp \bar{J}^-$,

$$\mathbf{i}_{2j-1},\, \bar{\mathbf{i}}_{2j-1} \;\; \text{are local integrals of motion}\,,$$
$$\boldsymbol{\beta}^*_{2j-1}, \boldsymbol{\gamma}^*_{2j-1}, \bar{\boldsymbol{\beta}}^*_{2j-1}, \bar{\boldsymbol{\gamma}}^*_{2j-1} \;\; \text{are fermions}\,.$$

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- 6 vertex model and expectation values (Theorems)

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- Scaling limit to CFT and sine-Gordon model (Conjectures)

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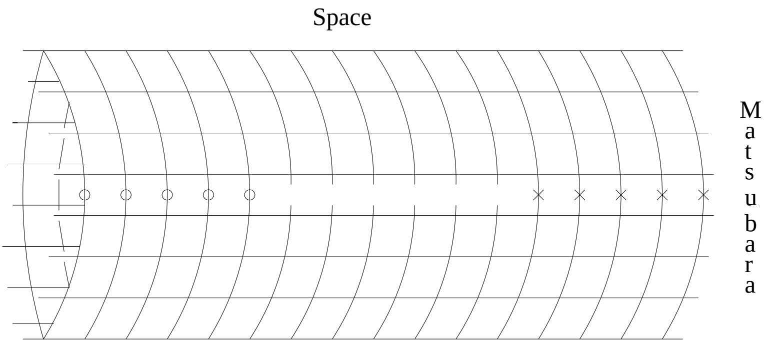
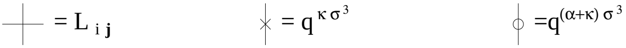
- Scaling limit to CFT and sine-Gordon model (Conjectures)
- Concluding remarks

Fermionic basis on the lattice

We consider a 6 vertex model on a cylinder





Analog of Φ_α : $q^{2\alpha S(0)}$, $\qquad S(0) = \frac{1}{2}\sum_{j=-\infty}^{0} \sigma_j^3$

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Space of quasi-local operators

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s\in\mathbb{Z}} \mathcal{W}_{\alpha-s,s},$$

$$\mathcal{W}_{\alpha-s,s} = \{q^{2(\alpha-s)S(0)}\mathcal{O} \mid \mathcal{O}\text{: local, spin } s\}$$

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$$Z_{\mathbf{n}}\Big\{q^{2\alpha S(0)}\mathcal{O}\Big\}=\frac{\langle\kappa+\alpha|\mathrm{Tr}_{\mathrm{S}}\Big(T_{\mathbf{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O}\Big)|\kappa\rangle}{\langle\kappa+\alpha|\mathrm{Tr}_{\mathrm{S}}\Big(T_{\mathbf{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\Big)|\kappa\rangle}\,,$$

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where

$$T_{\mathbf{S},\mathbf{M}} = \prod_{j=-\infty}^{\overset{\curvearrowright}{\infty}} T_{j,\mathbf{M}}(1)\,, \quad T_{j,\mathbf{M}}(\zeta) = \prod_{\mathbf{m}=\mathbf{1}}^{\overset{\curvearrowleft}{\mathbf{n}}} L_{j,\mathbf{m}}(\zeta)\,,$$

$$L_{j,\mathbf{m}}(\zeta) = q^{-\frac{1}{2}\sigma^3_j\sigma^3_{\mathbf{m}}} - \zeta^2 q^{\frac{1}{2}\sigma^3_j\sigma^3_{\mathbf{m}}} - \zeta(q-q^{-1})(\sigma^+_j\sigma^-_{\mathbf{m}} + \sigma^-_j\sigma^+_{\mathbf{m}})\,,$$

and $|\kappa\rangle$, $\langle\kappa|$ are eigen(co)vector of the transfer matirx

$$T_{\mathbf{M}}(\zeta,\kappa) = \mathrm{Tr}_j\Big[T_{j,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_j}\Big]\,.$$

Fermions on the lattice

One can construct fermions acting on $\mathcal{W}^{(\alpha)}$ (BJMST 2007–2009)

$$\mathbf{b}(\zeta),\ \mathbf{c}(\zeta),\ \mathbf{b}^*(\zeta),\ \mathbf{c}^*(\zeta)\,,$$

$$\mathbf{b}(\zeta)=\sum_{p=1}^{\infty}\mathbf{b}_p(\zeta^2-1)^{-p}\,,\quad \mathbf{b}^*(\zeta)=\sum_{p=1}^{\infty}\mathbf{b}^*_p(\zeta^2-1)^{p-1}\,,etc.$$

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such that

- They commute with integrals of motion: $\mathbf{t}^*_1,\mathbf{t}^*_2,\cdots$
- The following is a basis of $\mathcal{W}^{(\alpha)}$:

$$(\mathbf{t}^*_1)^p\mathbf{t}^*_{i_1}\cdots\mathbf{t}^*_{i_r}\mathbf{b}^*_{j_1}\cdots\mathbf{b}^*_{j_s}\mathbf{c}^*_{k_1}\cdots\mathbf{c}^*_{k_t}(q^{2\alpha S(0)})$$
$$(i_1\ge\cdots\ge i_r\ge 2,\ j_1>\cdots>j_s\ge 1,$$
$$k_1>\cdots>k_t\ge 1,\ p\in\mathbb{Z},\ r,s,t\ge 0)\,.$$

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NB: The fermions act not on states but on fields.
They have nothing to do with the Jordan-Wigner fermions.

The main formula

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$$Z_{\mathbf{n}}\left\{\mathbf{t}^*(\eta_1)\cdots\mathbf{t}^*(\eta_s)\mathbf{b}^*(\zeta_1)\cdots\mathbf{b}^*(\zeta_r)\mathbf{c}^*(\xi_r)\cdots\mathbf{c}^*(\xi_1)(q^{2\alpha S(0)})\right\}$$
$$=\prod_{i=1}^{s}2\rho(\eta_i)\cdot\det\left(\omega_{\mathbf{n}}(\zeta_j,\xi_k)\right)\,,$$

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$$=\prod_{i=1}^{s}2\rho(\eta_i)\cdot\det\left(\omega_{\mathbf{n}}(\zeta_j,\xi_k)\right)\,,$$

where

$$\rho(\eta)=\frac{T(\eta,\kappa+\alpha)}{T(\eta,\kappa)}\,,\quad T_{\mathbf{M}}(\eta,\kappa)|\kappa\rangle=T(\eta,\kappa)|\kappa\rangle\,,$$

and $\omega_{\mathbf{n}}(\zeta,\xi)$ is defined through the TBA data in the Matsubara space.

TBA data

- auxiliary function characterizing $|\kappa\rangle$

$$\log \mathfrak{a}(\zeta,\kappa) = -2\pi i\nu\kappa + \log\frac{a(\zeta)}{d(\zeta)} - \int_\gamma K_0(\zeta/\xi)\log(1+\mathfrak{a}(\xi,\kappa))\frac{d\xi^2}{\xi^2}\,,$$

where γ encircles the Bethe roots clockwise, and

$$a(\zeta) = (1-q\zeta^2)^{\mathbf{n}}\,,\quad d(\zeta) = (1-q^{-1}\zeta^2)^{\mathbf{n}}\,,$$

$$K_\alpha(\zeta) = \Delta_\zeta\psi(\zeta,\alpha)\,,\quad \psi(\zeta,\alpha) = \frac{\zeta^\alpha}{\zeta^2-1}\,,$$

$$\Delta_\zeta f(\zeta) = f(q\zeta) - f(q^{-1}\zeta)\,.$$

- resolvent

$$\mathcal{R}_{\mathrm{dress}} - \mathcal{R}_{\mathrm{dress}} \star K_\alpha = K_\alpha\,,$$

$$f\star g(\zeta,\xi) = \int_\gamma f(\zeta,\eta)g(\eta,\xi)\frac{1}{1+\mathfrak{a}(\eta,\kappa)}\frac{1}{\rho(\eta)}\frac{d\eta^2}{\eta^2}\,.$$

- formula for $\omega_{\mathbf{n}}$ is

$$\frac{1}{4}\omega_{\mathbf{n}}(\zeta,\xi) = f_{\mathrm{left}} \star (I + \mathcal{R}_{\mathrm{dress}}) \star f_{\mathrm{right}}(\zeta,\xi) - \omega_0(\zeta,\xi)\,,$$

where

$$f_{\mathrm{left}}(\zeta,\xi) = \frac{1}{2\pi i}\delta^-_\zeta \psi(\zeta/\xi,\alpha)\,, \quad f_{\mathrm{right}}(\zeta,\xi) = \delta^-_\xi \psi(\zeta/\xi,\alpha)\,,$$
$$\delta^-_\zeta f(\zeta) = f(q\zeta) - \rho(\zeta) f(\zeta)\,,$$
$$\omega_0(\zeta,\xi) = -\delta^-_\zeta \delta^-_\xi \Delta^{-1}_\zeta \psi(\zeta/\xi,\alpha)\,.$$

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- With a modification of $Z_{\mathbf{n}}$, κ, κ', α can be viewed as 'independent parameters'.

In the following analysis we make a technical assumption

$$\kappa' = \kappa\,.$$

CFT limit

In the limit

$$a = \frac{2\pi R}{\mathbf{n}} \to 0\,, \quad \zeta = (Ca)^{\nu}\lambda \quad (\lambda \text{ fixed})\,,$$

the Matsubara transfer matrix scales to that of chiral CFT (Bazhanov et al.1996)

$$T^{\mathrm{sc}}(\lambda,\kappa) = \lim T_{\mathbf{M}}\left((Ca)^{\nu}\lambda,\kappa\right)\,.$$

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$$T^{\mathrm{sc}}(\lambda,\kappa)=\lim T_{\mathbf{M}}\left((Ca)^{\nu}\lambda,\kappa\right)\,.$$

$T^{\mathrm{sc}}(\lambda,\kappa)$ is entire in λ^2 and has the asymptotics

$$\log T^{\mathrm{sc}}(\lambda,\kappa)\simeq\sum_{j=0}^{\infty}C_{2j-1}I_{2j-1}\lambda^{-\frac{2j-1}{\nu}}\quad(\lambda^2\to\infty)\,,$$

$$T^{\mathrm{sc}}(\lambda,\kappa)=2\cos\pi\nu\kappa+\sum_{j=1}^{\infty}G_{2j}\lambda^{2j}\quad(\lambda^2\to 0)\,.$$

The coefficients are

$$I_{2j-1} \ : \ \text{local integrals of motion}\,,$$
$$G_{2j} \ : \ \text{non-local integrals of motion}\,.$$

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has the asymptotics of the form

$$\begin{aligned}\omega^{\mathrm{sc}}(\lambda, \mu) &\simeq \sum_{j,k=1}^{\infty} \omega_{j,k}(\kappa, \alpha)\, \lambda^{-\frac{2j-1}{\nu}} \mu^{-\frac{2k-1}{\nu}} \quad (\lambda^2, \mu^2 \to \infty)\,, \\ &\simeq \sum_{j,k=1}^{\infty} \tilde{\omega}_{j,k}(\kappa, \alpha)\, \lambda^{-\frac{2j-1}{\nu}} \mu^{-\alpha+2k} \quad (\lambda^2 \to \infty, \mu^2 \to 0)\,.\end{aligned}$$

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We postulate that the coefficients in these expansions are three point functions of some local operators in CFT.

Conjecture(CFT)

- The following limit exists in the weak sense:

$$\boldsymbol{\beta}^*(\lambda) = \lim \frac{1}{2}\mathbf{b}^*\Big((Ca)^\nu\lambda\Big) \simeq \sum_{j=1}^{\infty} \boldsymbol{\beta}^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}\ ,$$

$$\boldsymbol{\gamma}^*(\lambda) = \lim \frac{1}{2}\mathbf{c}^*\Big((Ca)^\nu\lambda\Big) \simeq \sum_{j=1}^{\infty} \boldsymbol{\gamma}^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}\ .$$

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- The set $\{\beta^*_{I^+}\gamma^*_{I^-}\Phi_\alpha\}$ is a basis of the space of chiral descendants modulo integrals of motion.

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- The set $\{\boldsymbol{\beta}^*_{I^+}\boldsymbol{\gamma}^*_{I^-}\Phi_\alpha\}$ is a basis of the space of chiral descendants modulo integrals of motion.
- The Wick contraction rule applies with the pairing

$$\omega_{j,k}(\kappa,\alpha) = \frac{\langle 1-\kappa|\boldsymbol{\beta}^*_{2j-1}\boldsymbol{\gamma}^*_{2k-1}\Phi_\alpha|1+\kappa\rangle}{\langle 1-\kappa|\Phi_\alpha|1+\kappa\rangle}\,.$$

Example: The relations

$$\beta_1^*\gamma_1^*\Phi_\alpha = A\cdot \ell_{-2}\Phi_\alpha + B\cdot \ell_{-1}^2\Phi_\alpha\,,$$
$$\frac{\langle 1-\kappa|\beta_1^*\gamma_1^*\Phi_\alpha|1+\kappa\rangle}{\langle 1-\kappa|\Phi_\alpha|1+\kappa\rangle} = A\cdot\frac{\langle 1-\kappa|\ell_{-2}\Phi_\alpha|1+\kappa\rangle}{\langle 1-\kappa|\Phi_\alpha|1+\kappa\rangle} + B\cdot 0 \quad (\forall\kappa)$$

imply

$$A = D_1(\alpha)D_1(2-\alpha)\,,$$
$$D_a(\alpha) = \frac{1}{\sqrt{i\nu}}\Big(\Gamma(\nu)^{-1/\nu}\sqrt{1-\nu}\Big)^a\frac{\Gamma\Big(\frac{\alpha}{2}+\frac{a}{2\nu}\Big)}{\Gamma\Big(\frac{a+1}{2}\Big)\Gamma\Big(\frac{\alpha}{2}+\frac{1-\nu}{2\nu}a\Big)}\,.$$

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Similar (overdetermined) equations have been consistently solved upto degree 8.

Conjecture(screening operators)

The coefficients $\tilde{\omega}_{j,k}(\kappa,\alpha)$ are used to define the Wick pairing between $\boldsymbol{\beta}^*_{2j-1}$ and a new fermion $\boldsymbol{\gamma}^*_{\mathrm{screen},k}$.

Conjecture(screening operators)

The coefficients $\tilde{\omega}_{j,k}(\kappa,\alpha)$ are used to define the Wick pairing between β^*_{2j-1} and a new fermion $\gamma^*_{\mathrm{screen},k}$.

$$\boxed{\beta^*_{I^+}\gamma^*_{I^-}\Phi_{\alpha+2\frac{1-\nu}{\nu}} \simeq \tilde{C}_{1,0}(\alpha)\cdot\beta^*_{I^++2}\gamma^*_{I^--2}\beta^*_1\gamma^*_{\mathrm{screen},1}\Phi_\alpha\,.}$$

where

$$\tilde{C}_{1,0}(\alpha) = -\Gamma(\nu)^{2\alpha+2\frac{1-\nu}{\nu}}\frac{1}{2\nu}$$
$$\times\gamma\left(\frac{1}{2}-\frac{\alpha}{2}-\frac{1-\nu}{2\nu}\right)\gamma\left(\frac{\alpha}{2}+\frac{1-\nu}{2\nu}\right)\gamma\left(\nu-\nu\alpha\right)\,,$$
$$\gamma(x) = \Gamma(x)/\Gamma(1-x).$$

Scaling limit to sG model

Start with the 6 vertex model with alternating inhomogeneity parameters ζ_0, ζ_0^{-1} in both directions. The sine-Gordon model is obtained as a scaling limit

$$a = \frac{2\pi R}{\mathbf{n}} \to 0\,, \quad \zeta_0 = \left(\frac{4}{aM}\right)^{\nu} \to \infty\,,$$

with $R, M > 0$ fixed. ($M = M_{\mathrm{soliton}}$)
Repeating the same procedure (there are tricky points), one can introduce fermions

$$\boldsymbol{\beta}^*_{2j-1}, \quad \boldsymbol{\gamma}^*_{2j-1}, \quad \bar{\boldsymbol{\beta}}^*_{2j-1}, \quad \bar{\boldsymbol{\gamma}}^*_{2j-1}\,,$$
$$\boldsymbol{\beta}^*_{\mathrm{screen},j}, \quad \boldsymbol{\gamma}^*_{\mathrm{screen},j}, \quad \bar{\boldsymbol{\beta}}^*_{\mathrm{screen},j}, \quad \bar{\boldsymbol{\gamma}}^*_{\mathrm{screen},j}\,.$$

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For simplicity, we consider VEV in the case $R = \infty$.

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- For $\sharp J^+ = \sharp J^-$, $\sharp \bar{J}^+ = \sharp \bar{J}^-$ we have

$$
\frac{\langle \boldsymbol{\beta}^*_{J^+} \boldsymbol{\gamma}^*_{J^-} \bar{\boldsymbol{\beta}}^*_{\bar{J}^+} \bar{\boldsymbol{\gamma}}^*_{\bar{J}^-} \Phi_\alpha \rangle}{\langle \Phi_\alpha \rangle} = \boldsymbol{\mu}^{2\nu^{-1}(\sum_{a\in J^+} a + \sum_{b \in J^+} b)}
$$

$$
\times \prod_{a \in J^+} \frac{i}{\nu} \cot \frac{\pi}{2\nu}(a + \nu\alpha) \prod_{b \in J^-} \frac{i}{\nu} \cot \frac{\pi}{2\nu}(b - \nu\alpha) \, .
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$$\times\prod_{a\in J^+}\frac{i}{\nu}\cot\frac{\pi}{2\nu}(a+\nu\alpha)\prod_{b\in J^-}\frac{i}{\nu}\cot\frac{\pi}{2\nu}(b-\nu\alpha)\,.$$

- Under the shift $\alpha\to\alpha+2(1-\nu)/\nu$:

$$\langle\Phi_{\alpha+2\frac{1-\nu}{\nu}}\rangle=\langle\Phi_\alpha\rangle\tilde{C}_{1,0}(\alpha)\boldsymbol{\mu}^{2/\nu-2(2-\alpha)}$$
$$\times\frac{1}{\nu}\cot\frac{\pi}{2\nu}(1+\nu\alpha)\tan\frac{\pi\nu}{2}(2-\alpha)\,.$$

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- For $R < \infty$ (Zamolodchikov, 2004)

$$\langle T_{z,z} T_{\bar{z},\bar{z}} \rangle = \langle T_{z,z} \rangle \langle T_{\bar{z},\bar{z}} \rangle - \langle T_{z,\bar{z}} \rangle^2$$

Form factors

$$f_{\mathcal{O}}^{(2n)}(\theta_1, \cdots, \theta_{2n}) = \langle \theta_1, \cdots, \theta_{2n} | \mathcal{O} | \mathrm{vac} \rangle$$

Solutions to the form factor bootstrap equations are given by integrals (Smirnov, 1986)

$$\begin{aligned} f_{\mathcal{O}_\alpha}^{(2n)}(\theta_1, \cdots, \theta_{2n}) &= \langle \Phi_\alpha \rangle \zeta(\theta_1, \cdots, \theta_{2n}) \\ &\times \sum_{\epsilon_1, \cdots, \epsilon_{2n}} w_{\epsilon_1, \cdots, \epsilon_{2n}}(\theta_1, \cdots, \theta_{2n}) \mathcal{F}_{\mathcal{O}_\alpha}^{(2n)}(\theta_{I^-} | \theta_{I^+}) \\ \mathcal{F}_{\mathcal{O}_\alpha}^{(2n)}(\theta_{I^-} | \theta_{I^+}) &= \int d^n \sigma \chi_\alpha(\sigma_1, \cdots, \sigma_n) \ell_{I^+ \sqcup I^-}^{(n)}(\sigma_1, \cdots, \sigma_n) \\ &\times L_{\mathcal{O}_\alpha}^{(n)}(S_1, \cdots, S_n) \end{aligned}$$

where $S_j = e^{\sigma_j}$.

$L^{(n)}_{\mathcal{O}_\alpha}(S_1,\cdots,S_n|B_1,\cdots,B_{2n})$ is a Laurent polynomial such that

- anti-symmetric in S_j, symmetric in B_i
- recurrence relation

$$L^{(n)}_{\mathcal{O}}\Big|_{B_{2n-1}=B_{2n}=S_n} = S_n\prod_{i=1}^{n-1}(S_i^2-S_n^2)\cdot L^{(n-1)}_{\mathcal{O}}(S_1,\cdots,S_{n-1}|B_1,\cdots,B_2$$

For the primary field:

$$L^{(n)}_{\Phi_\alpha}(S_1,\cdots,S_n) = \prod_{p=1}^{n} S_p\prod_{p>r}(S_p^2-S_r^2)\,.$$

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Unexpectedly, we have found that, at the level of form factors, our fermions coincide with BBS fermions.

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- Conceptual understanding of fermions
- Understanding $\omega(\zeta, \xi)$

THANK YOU VERY MUCH FOR YOUR ATTENTION