CFT AND INTEGRABLE MODELS

8-th Workshop, Bologna, September 12-14, 2011

Solving the AdS/CFT Y-system

Vladimir Kazakov (ENS,Paris) **with Gromov, Leurent, Volin**
(to appear)

Integrability in AdS/CFT

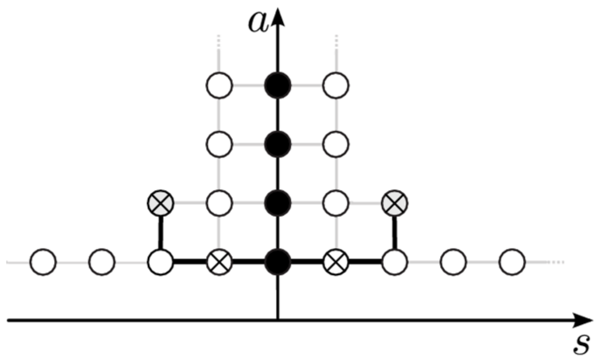
- Integrable planar superconformal 4D N=4 SYM and 3D N=8 Chern-Simons... (non-BPS, summing genuine 4D Feynman diagrams!)

- Based on AdS/CFT duality to very special 2D superstring σ-models on AdS-background

- Y-system (for planar AdS_5/CFT_4 , AdS_4/CFT_3 ,...) calculates exact anomalous dimensions of all local operators at any coupling

- Y-system is an infinite set of functional or integral nonlinear eqs.

$$Y_{a,s}\left(u+\frac{i}{2}\right)\, Y_{a,s}\left(u-\frac{i}{2}\right) = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\,\left[1+\frac{1}{Y_{a+1,s}}\right]}$$



Gromov,V.K.,Vieira

- Problem: how to transform Y-system into a finite system of non-linear integral equations (FiNLIE) using its Hirota discrete integrable dynamics and analyticity properties in spectral parameter?

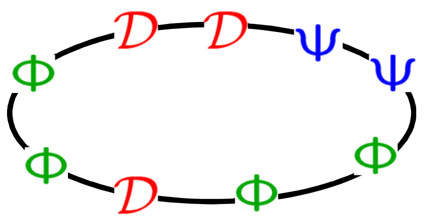
CFT: N=4 SYM as a superconformal 4D QFT

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x\, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

- 4D superconformal QFT! Global symmetry PSU(2,2|4)
- Operators in 4D

$$\mathcal{O}(x) = \mathrm{Tr}\,[\overbrace{\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots}^{L\,=\,\#\mathrm{bosons}\,+\,\#\mathrm{fermions}}](x)$$

$$+\mathrm{permutations}$$



- 4D Correlators:

scaling dimensions, structure constants: non-trivial functions of 'tHooft coupling λ!

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{\delta_{ij}}{|x|^{2\Delta_j(\lambda)}}$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}(\lambda)}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_i+\Delta_k-\Delta_j}}$$

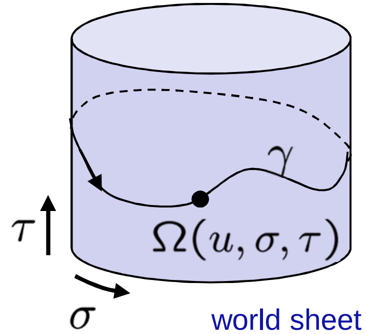
They describe the whole conformal theory via operator product expansion

*

AdS-dual classical superstring and its algebraic curve

- 2D σ-model on a coset $\dfrac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

- String equations of motion and constraints can be recasted into zero curvature condition

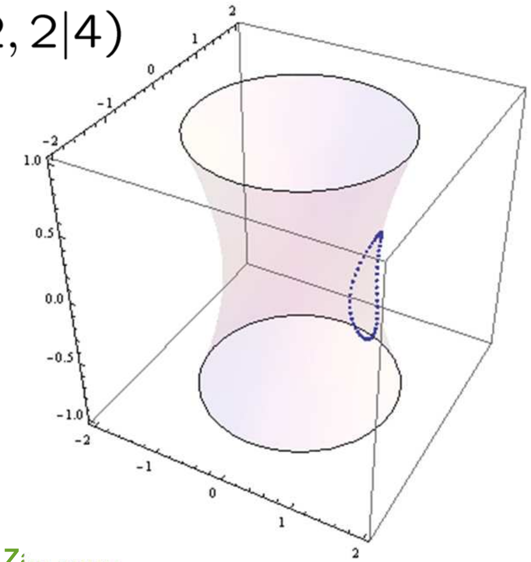


$$(d+\mathcal{A}(u))\wedge(d+\mathcal{A}(u))=0,$$

Zakharov,Mikhailov
Bena,Roiban,Polchinski

- Monodromy matrix $\Omega(u)=P\exp\oint_\gamma \mathcal{A}(u) \quad \in PSU(2,2|4)$

  encodes infinitely many conservation lows



- Eigenvalues – conserved quantities

$$\Omega(u)\;=\;\{e^{i\tilde{p}_1(u)},e^{i\tilde{p}_2(u)},e^{i\tilde{p}_3(u)},e^{i\tilde{p}_4(u)}||e^{i\hat{p}_1(u)},e^{i\hat{p}_2(u)},e^{i\hat{p}_3(u)},e^{i\hat{p}_4(u)}\}$$

- Algebraic curve for quasi-momenta:

$$\mathcal{P}(p,u)=\mathrm{sdet}\left(\mathbf{I}-e^{-ip}\cdot\Omega(u)\right)\sim\frac{0}{0}$$

V.K.,Marshakov,Minahan,Za[illegible]
Beisert,V.K.,Sakai,Zarembo

- Dimension of YM operator $\Delta_A(\lambda)$ **=** Energy of a string state

Classical $\mathbb{Z}_4$ symmetry and characters

- Trace of classical monodromy matrix is a psu(2,2|4) character. We take it in irreps for $a \times s$ rectangular Young tableaux:

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$$T_{a,s}(u) = \mathrm{Tr}_{a,s}\,\Omega(u)$$

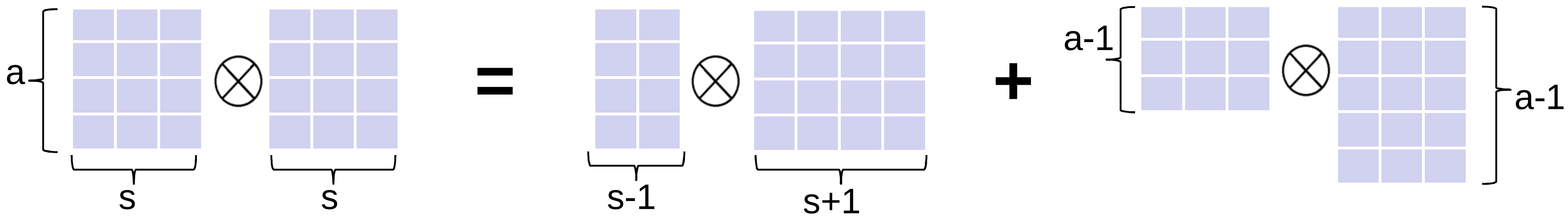
- Classical $\mathbb{Z}_4$ is an authomorphism of the string coset. It induces a certain monodromy on the algebraic curve: for a special path $\gamma(u)$ surrounding a branch point we have:

$$\begin{aligned} T_{a,s}(u) &= (-1)^s T^c_{a,-s}(u^*), \quad u^* = \gamma(u), && \text{if } |s| \geq a \\ T_{a,s}(u) &= (-1)^a T^c_{-a,s}(u^*) \quad u^* = \gamma(u), && \text{if } a \geq |s| \end{aligned}$$

- We will generalized this $\mathbb{Z}_4$ property to the full quantum case!

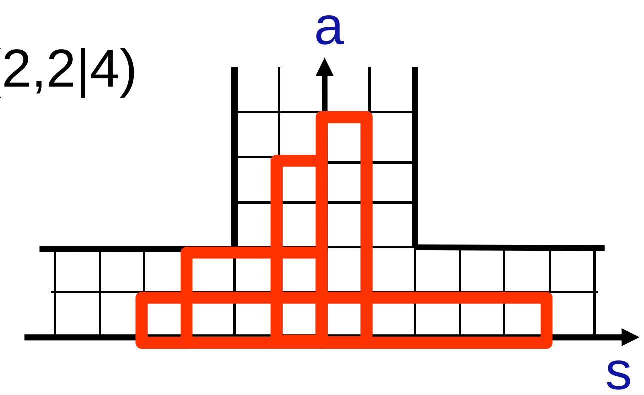
(Super-)group theoretical origins of Y-system

- A curious property of gl(N|M) representations with rectangular Young tableaux:



- For characters – simplified Hirota eq.: $$T_{a,s}^2 = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$

- Boundary conditions for Hirota eq.:
  ∞ - dim. unitary highest weight representations of u(2,2|4) in “T-hook” !



Kwon
Cheng,Lam,Zhang

Gromov, V.K., Tsuboi

- Solution of Hirota for any irrep: Jacobi-Trudi formula for GL(K|M) characters:

$$T_{a,s}[g] = \det_{1\le i,j\le a} T_{1,s-i+j}[g]\,, \qquad g \in GL(K|M).$$

Quantisation of characters and AdS/CFT T-system

- How to quantize?
  - Trust the integrability!
  - Use experience from known sigma-models

- Hirota equation for characters promoted to the full equation for T-functions ("transfer matrices"):

$$T_{a,s}\left(u+\frac{i}{2}\right)T_{a,s}\left(u-\frac{i}{2}\right)=T_{a,s-1}(u)\,T_{a,s+1}(u)+T_{a+1,s}(u)T_{a-1,s}(u)$$

- Classical limit: highly excited long strings/operators, strong coupling:

$$L\sim g\sim u\to\infty$$

- Only slow parametric dependence of characters on spectral parameter:

$$T_{a,s}\left(u+\cancel{\frac{i}{2g}}\right)T_{a,s}\left(u-\cancel{\frac{i}{2g}}\right)=T_{a,s-1}(u)\,T_{a,s+1}(u)+T_{a+1,s}(u)T_{a-1,s}(u)$$

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Discrete "classical" integrable dynamics!

Y-system and Hirota eq.: discrete integrable dynamics

- By the change of variables $$Y_{a,s} = \frac{T_{a,s+1}\, T_{a,s-1}}{T_{a+1,s}\, T_{a-1,s}}$$

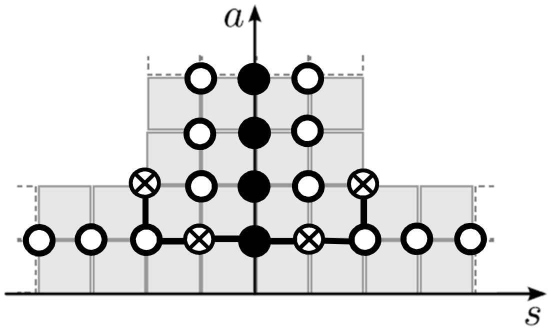
T-system (Hirota equation) reduces to Y system

$$\frac{Y_{a,s}^{+}\, Y_{a,s}^{-}}{Y_{a+1,s}\, Y_{a+1,s}} = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\,\left[1+Y_{a+1,s}\right]}$$

definitions:

$$f^{[\pm a]} := f(u \pm ia/2)$$

$$f^{\pm} := f(u \pm i/2)$$



Hirota eq. in T-hook for AdS/CFT

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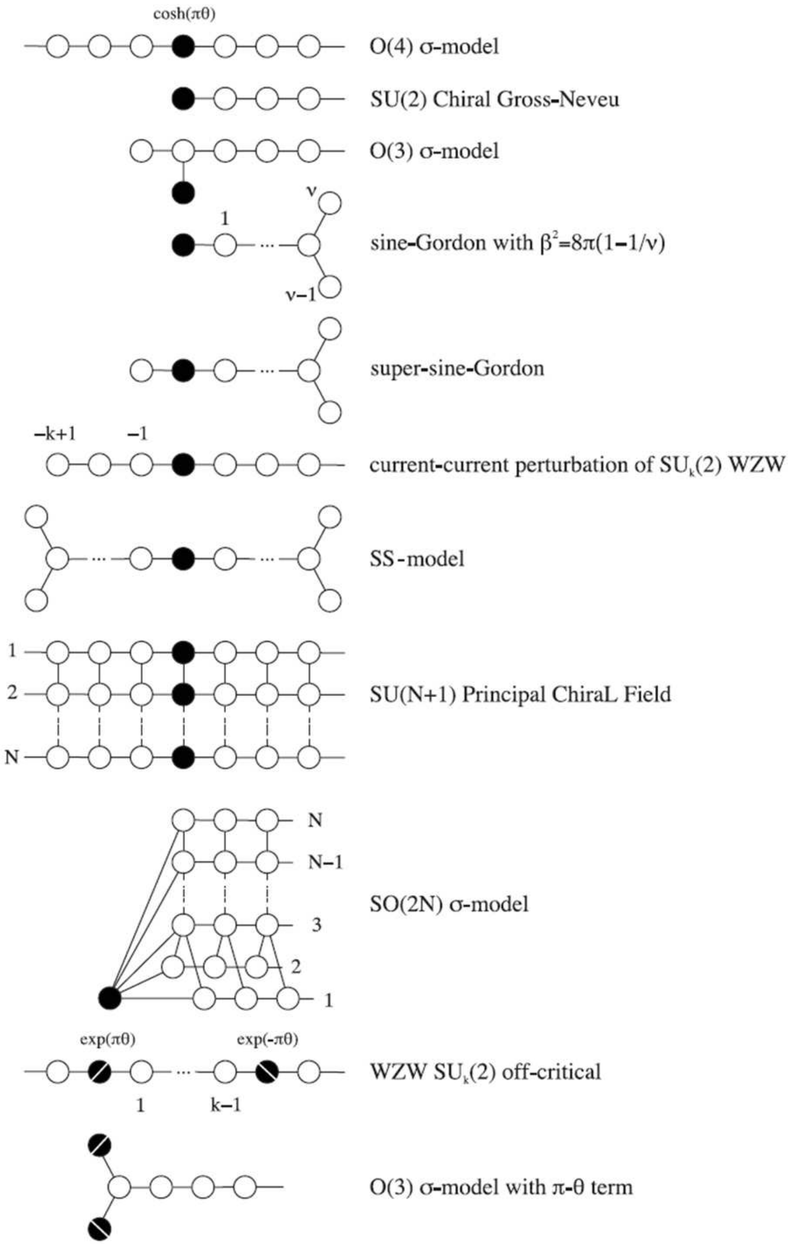
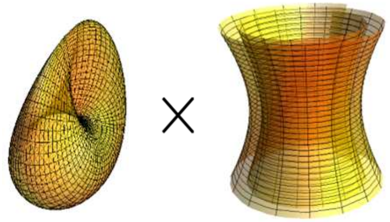
- Gauge symmetry of Hirota equation living Y-functions invariant

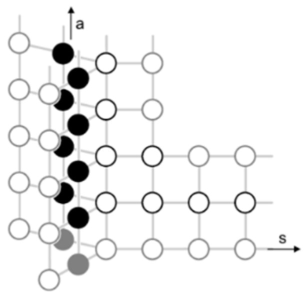
$$T_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[-a+s]} g_4^{[-a-s]}\; T_{a,s}$$

Y-systems for other σ-models



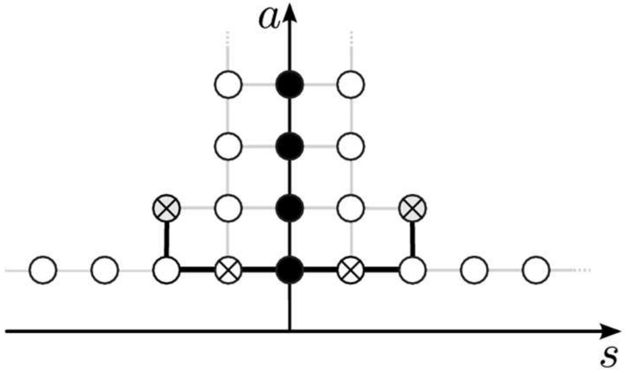
3d ABJM model: CP^3 x AdS_4, …





Gromov,V.K.,Vieira
Bombardelli,Fiorvanti,Tateo
Gromov,Levkovich-Maslyuk

AdS/CFT Y-system and asymptotics



• Large L asymptotics:

$$Y_{a,s}(u) \simeq C_{a,s} e^{\delta_{s,0}\, L\, i p_a(u)}$$

• Momentum of elementary excitation $\quad p_a(u) = -i \log \frac{x^{[+a]}}{x^{[-a]}}$

- zero mode of discrete D'Alembert operator in the l.h.s. of Y-system

We should fix it and find the corresponding energy as well.

• TBA equations can be induced from Y-system with certain analyticity assumptions

$$\log Y_{a,s} = L\,\delta_{a,0} \frac{\partial}{\partial u}\tilde{\epsilon}_a + \sum_{a',s'} K_{a,s;a',s'} * \log\left(1 + Y_{a',s'}(u)\right)$$

Bombardelli, Fiorvanti, Tateo
Gromov,V.K.,Vieira
Arutyunov, Frolov

Dispersion relation in physical and crossing channels

- Exact one particle dispersion relation: $\epsilon^2 = 1 + \lambda \sin^2 \frac{p}{2}$

Santambrogio,Zanon
Beisert,Dippel,Staudacher
N.Dorey

- Bound states (fusion) $\epsilon_a^2 = a^2 + \lambda \sin^2 \frac{p_a}{2}$

$$\text{—}Z\text{—}Z\text{—}\underset{u-\frac{i}{2}}{\mathcal{D}}\text{—}Z\text{—}Z\text{—}\underset{u+\frac{i}{2}}{\mathcal{D}}\text{—}Z\text{—}$$

- Changing physical L-circle to cross channel R-circle (Alyosha's trick!)

$$\epsilon_a^2 = a^2 + \lambda \sin^2 \frac{p_a}{2} \quad \Rightarrow \quad -\tilde{\epsilon}_a^2 = a^2 - \lambda \sinh^2 \frac{\tilde{p}_a}{2}$$

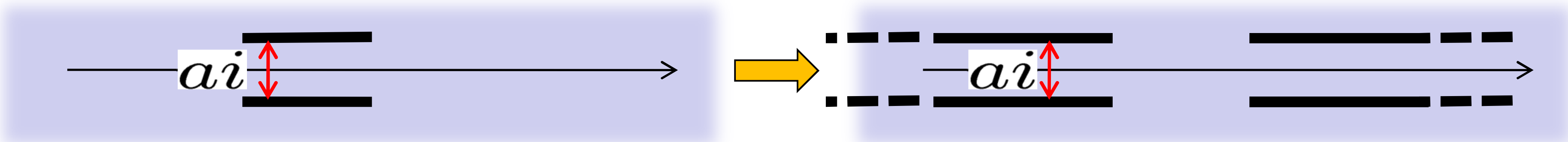
Ambjorn,Janik,Kristjansen
Arutyunov,Frolov

- Parametrization for the dispersion relation by Zhukovsky map: $u = \sqrt{\lambda}\left(z + \frac{1}{z}\right)$

$$\begin{cases} p_a(u) = \frac{1}{i} \log \frac{z(u + ia/2)}{z(u - ia/2)} \\ \epsilon_a(u) = 2i\sqrt{\lambda}\,[z(u - ia/2) - z(u + ia/2)] + 1 \end{cases}$$

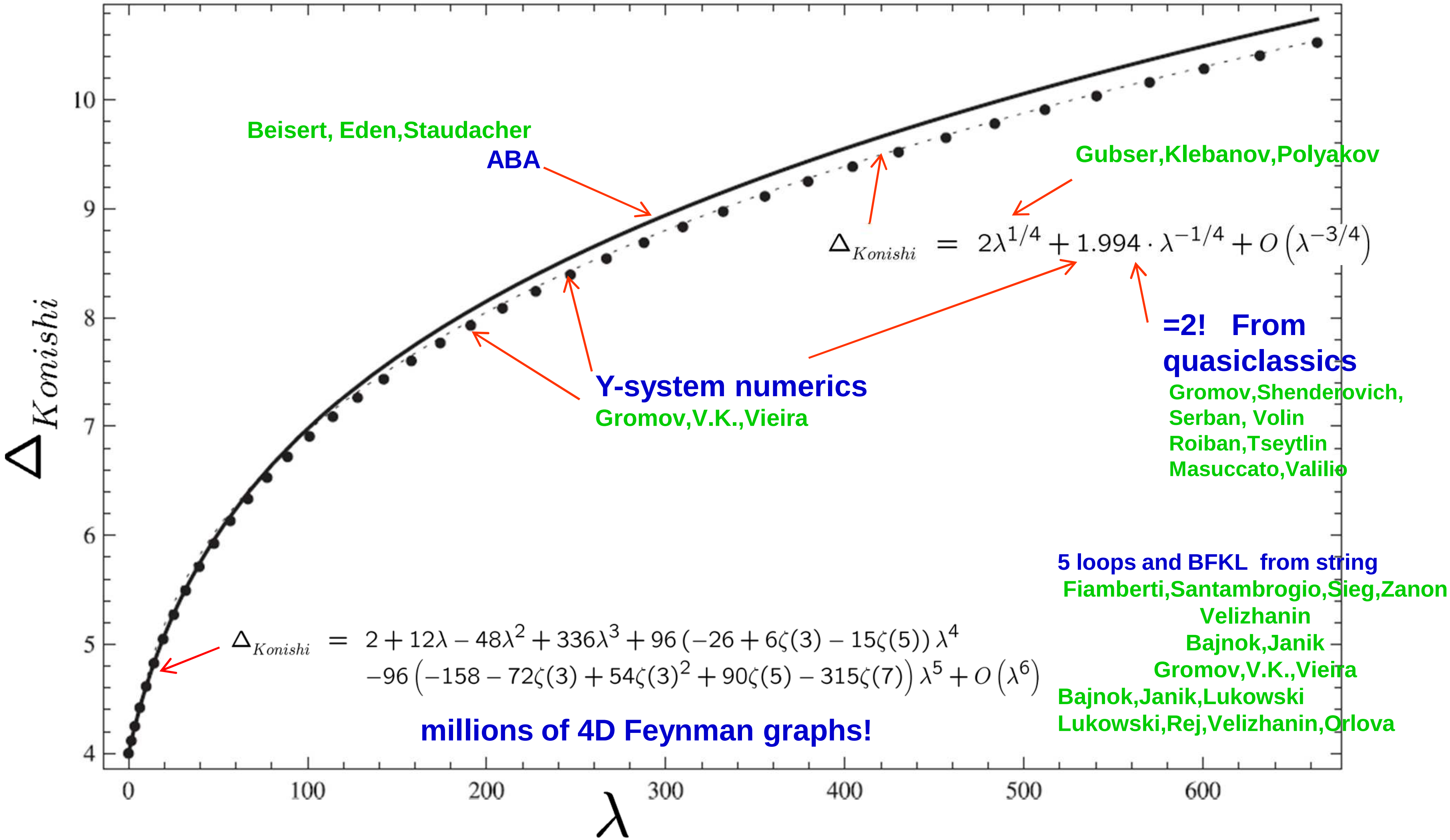
- From physical to crossing kinematics: continuation through the cut

$$z = \frac{1}{2\sqrt{\lambda}}\left(u + \sqrt{u - 2\sqrt{\lambda}}\sqrt{u + 2\sqrt{\lambda}}\right) \quad \Rightarrow \quad \tilde{z} = \frac{1}{2\sqrt{\lambda}}\left(u + i\sqrt{4\lambda - u^2}\right)$$



cuts in complex u -plane

Konishi operator $\mathrm{Tr}\ [\mathcal{D}, Z]^2$: numerics from **Y-system**

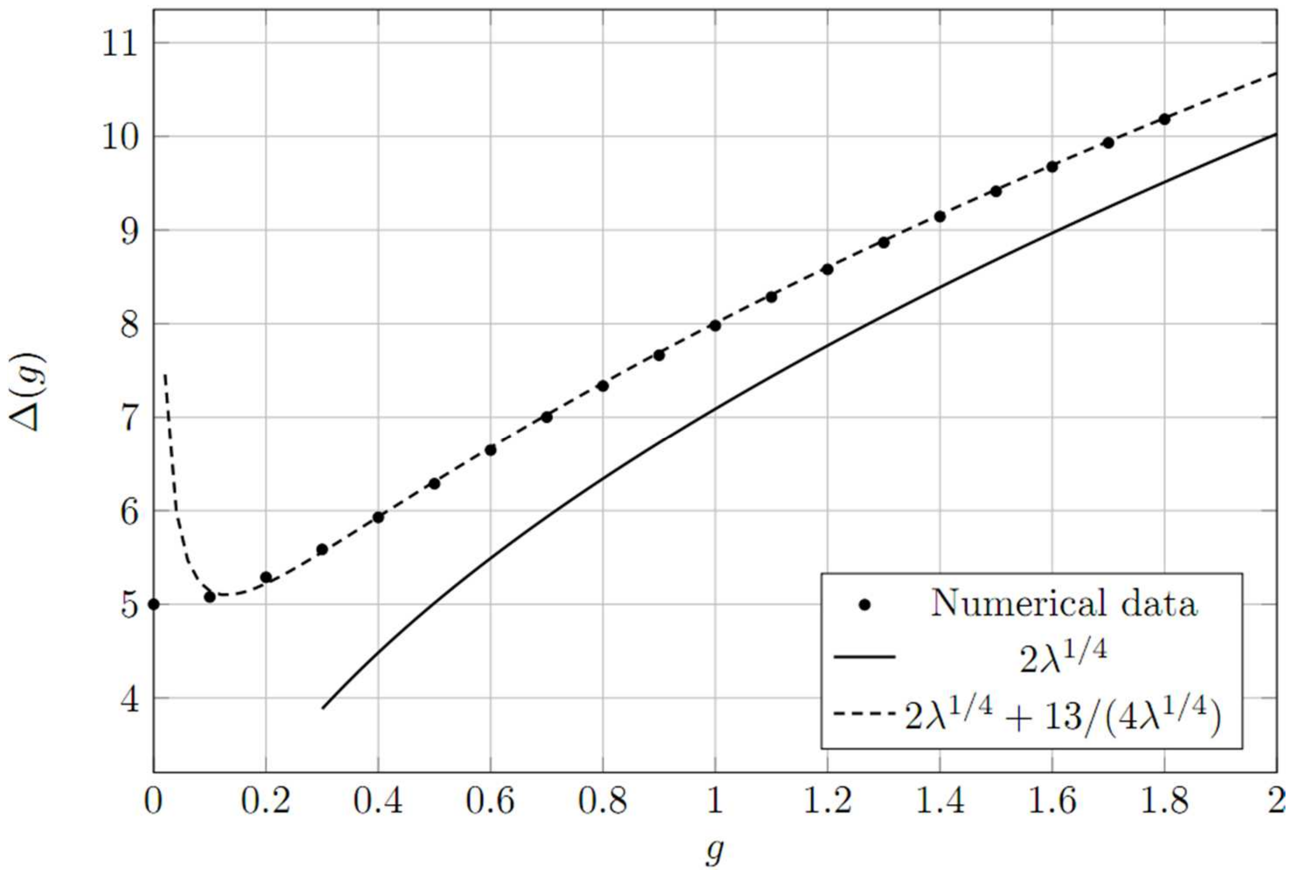


- Y-system passes all known tests

Numerics for $\mathrm{Tr}\,\mathcal{D}^S Z^J$

Gromov, Shenderovich, Serban, Volin

Also works for $S = 2, \quad J = 3, \quad n = 1$

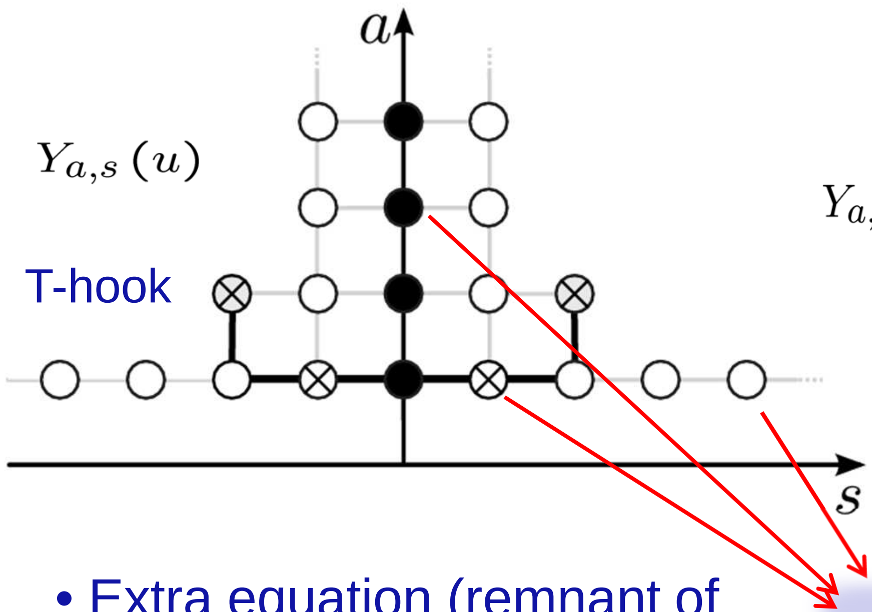


Y-system for excited states of AdS/CFT at finite size

Gromov,V.K.,Vieira

$Y_{a,s}(u)$

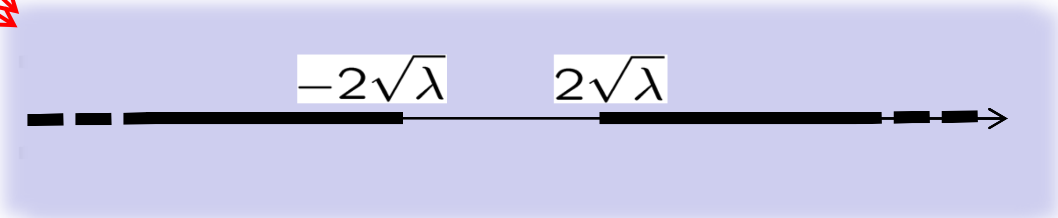
T-hook



$$Y_{a,s}\left(u+\frac{i}{2}\right) Y_{a,s}\left(u-\frac{i}{2}\right) = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\left[1+\frac{1}{Y_{a+1,s}}\right]}$$

- Complicated analyticity structure in u dictated by non-relativistic dispersion

cuts in complex u-plane



- Extra equation (remnant of classical Z_4 monodromy):

$$Y_{2,\pm 2}(u+i0) \cdot Y_{1,\pm 1}(u-i0) = 1$$

- Energy : (anomalous dimension)

$$\Delta - \Delta_0 = f_{min} = \int \frac{du}{2\pi i}\, \partial_u \tilde{\epsilon}_a \, \log\left(1+Y_{a,0}\right)$$

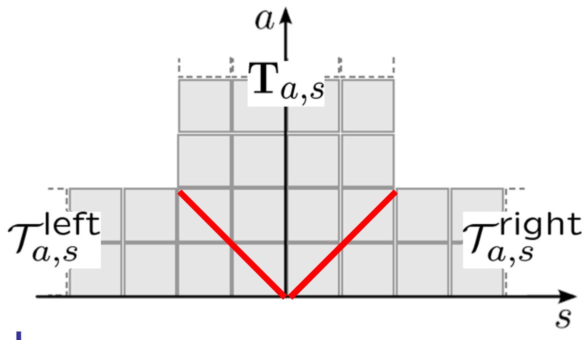
- u_j obey the exact Bethe eq.: $\quad Y_{1,0}(u_j)+1=0$

New insights into analyticity:
Cavaglia,Fioravanti,Tateo
Hegedus,Balog

Solution of AdS/CFT T-system in terms of finite number of non-linear integral equations

Gromov,V.K.,Leurent,Volin
(to appear)

- No single analyticity friendly gauge
  for T-functions of right, left and upper bands.



- Firstly, we should parameterize T's in all three bands
  in three different, analyticity friendly gauges,
  also respecting their reality and some other basic properties

- Secondly, we should find gauge transformations

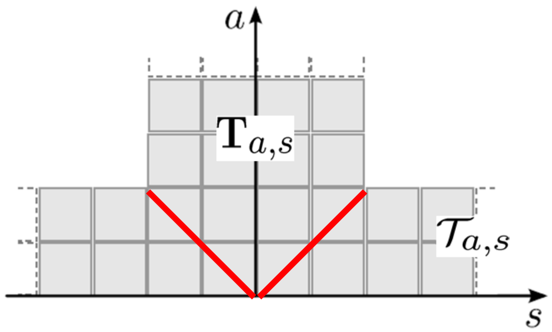
$$\mathbf{T}_{a,s} = g_1^{[a+s]} g_2^{[a-s]} g_3^{[-a+s]} g_4^{[-a-s]} \ \mathcal{T}_{a,s}$$

relating $\mathbf{T}_{a,s}$ to $\mathcal{T}_{a,s}^{\mathrm{right}}$ and $\mathcal{T}_{a,s}^{\mathrm{left}}$

- The very existence of such a gauge transform
  almost completely fixes the solution!
  (as in principal chiral field model!) Gromov, V.K., Vieira
  V.K., Leurent

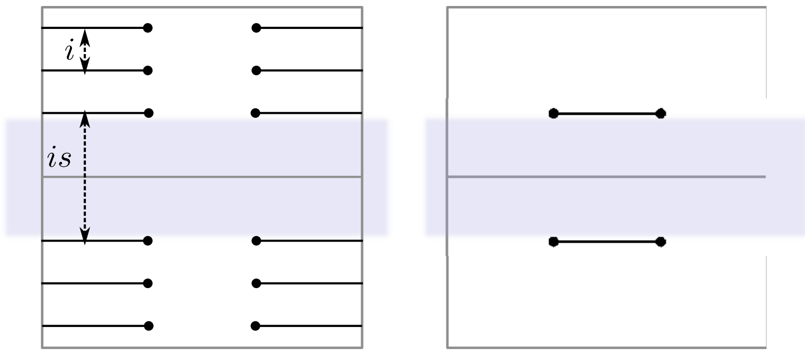
Magic sheet and solution for the right band

- T-functions have certain analyticity strips $\mathcal{A}_s$
  (between two closest to $\mathbb{R}$ Zhukovsky cuts)
  $\mathcal{T}_{0,\pm s} = 1$, $\mathcal{T}_{1,\pm s} \in \mathcal{A}_s$, $\mathcal{T}_{2,\pm s} \in \mathcal{A}_{s-1}$



- Original T-system is in mirror sheet (long cuts)
- The property $Y_{2,\pm 2}(u+i0) \cdot Y_{1,\pm 1}(u-i0) = 1$
  suggests that the functions $\mathcal{T}_{a,s}$
  are much simpler on the “magic” sheet – with only short cuts:

$\mathcal{A}_s$



- Only two cuts left on the magic sheet for $\mathcal{T}_{1,s}$!

$$\widehat{\mathcal{T}}_{0,s} = 1 \,,$$

$$\widehat{\mathcal{T}}_{1,s} = q^{[+s]} - q^{[-s]}, \qquad q = S_{M-1}(u) + \frac{1}{2\pi i}\int_{-2g}^{2g} \frac{\rho(v)dv}{u-v},$$

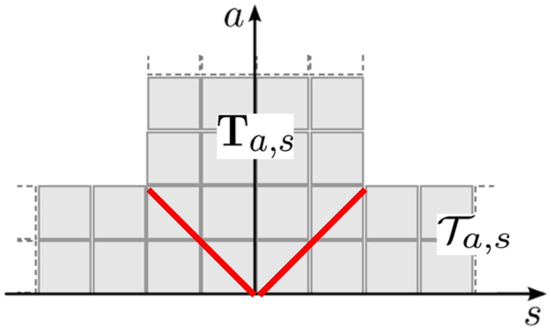
$$\widehat{\mathcal{T}}_{2,s} = \widehat{\mathcal{T}}_{1,1}^{[+s]}\widehat{\mathcal{T}}_{1,1}^{[-s]}$$

- Right band parameterized by one density (and a polynomial)

Magic sheet for the upper band

- Analyticity strips (dictated by Y-functions)

$$\mathbf{T}_{a,0} \in \mathcal{A}_{a+1} \quad , \quad \mathbf{T}_{a,\pm 1} \in \mathcal{A}_{a} \quad , \quad \mathbf{T}_{a,\pm 2} \in \mathcal{A}_{a-1}$$



- The property $Y_{2,\pm 2}(u+i0) \cdot Y_{1,\pm 1}(u-i0) = 1$ again suggests to go to magic sheet (short cuts), but the solution is more complicated.

- The irreps n^2 and 2^n are in fact the same typical irrep so it is natural to impose for our physical gauge

$$\mathbf{T}_{n,2} = \mathbf{T}_{2,n}$$

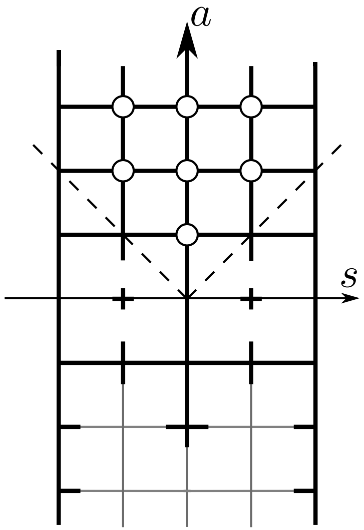
- From the unimodularity of the quantum monodromy matrix it follows that the function $\mathbf{T}_{0,0}$ is i-periodic

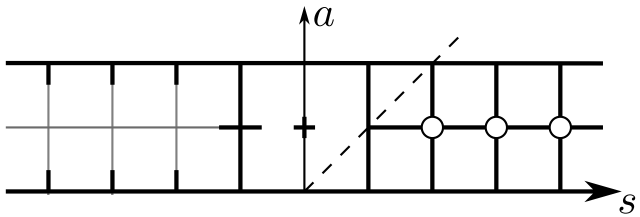
$$\mathbf{T}_{0,0}^{+} = \mathbf{T}_{0,0}^{-}$$

Quantum $\mathbb{Z}_4$ symmetry

- $T_{a,s}$ can be analytically continued in labels a, s

$$\hat{\mathbf{T}}_{a,s} = (-1)^s \hat{\mathbf{T}}_{-a,s}\,, \qquad \hat{\mathcal{T}}_{a,s} = (-1)^a \hat{\mathcal{T}}_{a,-s}\,.$$





- Analytically continued $\hat{\mathbf{T}}_{a,s}$ and $\hat{\mathcal{T}}_{a,s}$ satisfy the Hirota equations, each in its infinite strip.

Wronskian solution and parameterization for the upper band

- Use Wronskian formula for general solution of Hirota in a band of width N

Krichever,Wiegmann, Lipan, Zabrodin

- From reality, $\mathbb{Z}_4$ symmetry and asymptotic propertie at large L and considering only left-right symmetric states $\hat{\mathbf{T}}_{a,-s} = \hat{\mathbf{T}}_{a,s}$ we find

$$
\begin{aligned}
\mathsf{T}_{a,\pm 1} &= q_1^{[+a]}\bar{q}_2^{[-a]} + q_2^{[+a]}\bar{q}_1^{[-a]} + q_3^{[+a]}\bar{q}_4^{[-a]} + q_4^{[+a]}\bar{q}_3^{[-a]}\,, \\
\mathsf{T}_{a,0} &= q_{12}^{[+a]}\bar{q}_{12}^{[-a]} + q_{34}^{[+a]}\bar{q}_{34}^{[-a]} - q_{14}^{[+a]}\bar{q}_{14}^{[-a]} - q_{23}^{[+a]}\bar{q}_{23}^{[-a]} - q_{13}^{[+a]}\bar{q}_{24}^{[-a]} - q_{24}^{[+a]}\bar{q}_{13}^{[-a]}\,, \\
\mathsf{T}_{a,-1} &= \left(U^{[+a]}\bar{U}^{[-a]}\right)^2 \mathsf{T}_{a,1}\,.
\end{aligned}
$$

Gromov,V.K. Leurent, Tsuboi

- Plucker relations

$$
\begin{aligned}
q_\emptyset &= q_i^+ q_j^- - q_j^+ q_i^-\,, \\
q_{ijk} q_i &= q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^-\,.
\end{aligned}
$$

- Relation between $\mathbf{T}_{a,s}$ and $\mathsf{T}_{a,s}$

$$
\mathbf{T}_{a,s} = \mathsf{T}_{a,s}\, f^{[a+s]} f^{[a-s]} \bar{f}^{[-a-s]} \bar{f}^{[-a+s]} \left(U^{[+a]}\bar{U}^{[-a]}\right)^{[s]_D}
$$

Parameterization of the upper band: continuation

- Our choice of parameterization in terms of a spectral density ρ_2
  left-right wing exchange function U
  and two polynomials encoding Bethe roots

$$q_1 = 1, \qquad q_2 = P(u) - \int_{-\infty}^{\infty} \frac{dv}{2\pi i} \frac{\rho_2(v)}{u-v}, \qquad q_{12} = \tilde{Q} \,,$$

- Remarkably, since q_1, q_2 are analytic in the upper half plane,
  and q_{12} analytic above $-i/2 + \mathbb{R}$
  all T-functions have the right analyticity strips!
  $\mathbb{Z}_4$ symmetry is also respected

Relating right(left) and upper bands

$$\mathbb{T}_{a,s} = \mathbf{T}_{a,s}(\mathbf{T}_{0,0}^{[a+s]})^{\frac{a-2}{2}}$$

here we gauge-transformed

$$\begin{aligned}
\widehat{\mathbb{T}}_{0,s} &= 1 \\
\widehat{\mathbb{T}}_{1,s} &= \hat{h}^{[+s]}\hat{h}^{[-s]}\widehat{\mathcal{T}}_{1,s} \\
\widehat{\mathbb{T}}_{2,s} &= \hat{h}^{[+s+1]}\hat{h}^{[+s-1]}\hat{h}^{[-s+1]}\hat{h}^{[-s-1]}\widehat{\mathcal{T}}_{2,s}
\end{aligned}$$

Closing FiNLIE: sawing together 3 bands

- From Hirota and analyticity - closed system of equations – FiNLIE
- Strategy: relate all unknown functions of upper and right bands to $Y_{1,1},\ Y_{2,2}$
- Then get from analyticity of T's an eq. for $Y_{1,1},\ Y_{2,2}$

- Expressing right band density ρ through $Y_{1,1},\ Y_{2,2}$

$$\frac{1+Y_{1,1}}{1+1/Y_{2,2}} = \frac{\mathcal{T}^{+}_{1,1}\mathcal{T}^{-}_{1,1}\mathcal{T}_{2,3}}{\mathcal{T}^{+}_{2,2}\mathcal{T}^{-}_{2,2}\mathcal{T}_{0,1}}$$

- Expressing upper band density ρ_2 through $Y_{1,1},\ Y_{2,2}$

$$\frac{1+Y_{2,2}}{1+1/Y_{1,1}} = \frac{\mathsf{T}^{+}_{2,2}\mathsf{T}^{-}_{2,2}\mathsf{T}_{1,0}}{\mathsf{T}^{+}_{1,1}\mathsf{T}^{-}_{1,1}\mathsf{T}_{3,2}}$$

Closing FiNLIE: equations for U and $Y_{1,1},\ Y_{2,2}$

- We get spectral representations of $Y_{1,1},\ Y_{2,2}$ from analytic properties of

$$\left(\frac{f^-}{f^+}\right)^2 = \frac{\mathbf{T}^-_{0,0}}{\mathbf{T}_{1,0}}\frac{\mathsf{T}_{1,0}}{\mathsf{T}^-_{0,0}} = \frac{1}{Y_{1,1}Y_{2,2}}\frac{\mathsf{T}_{1,0}}{\mathsf{T}^-_{0,0}}$$

- Spectral representations for U from analytic properties of

$$\left(\frac{U}{U^{[2]}}\frac{f^{[1]}\widehat{h}^{[2]}}{f^{[3]}\widehat{h}}\right)^2 = \frac{Y_{1,1}}{Y_{2,2}}\frac{\mathsf{T}^-_{0,0}}{\mathsf{T}_{1,0}}\left(\frac{\mathsf{T}_{2,1}}{\mathcal{T}_{1,2}}\frac{\mathcal{T}^-_{1,1}}{\mathsf{T}^-_{1,1}}\right)^2$$

Both analytic in the upper half plane and approaching const. at infinity

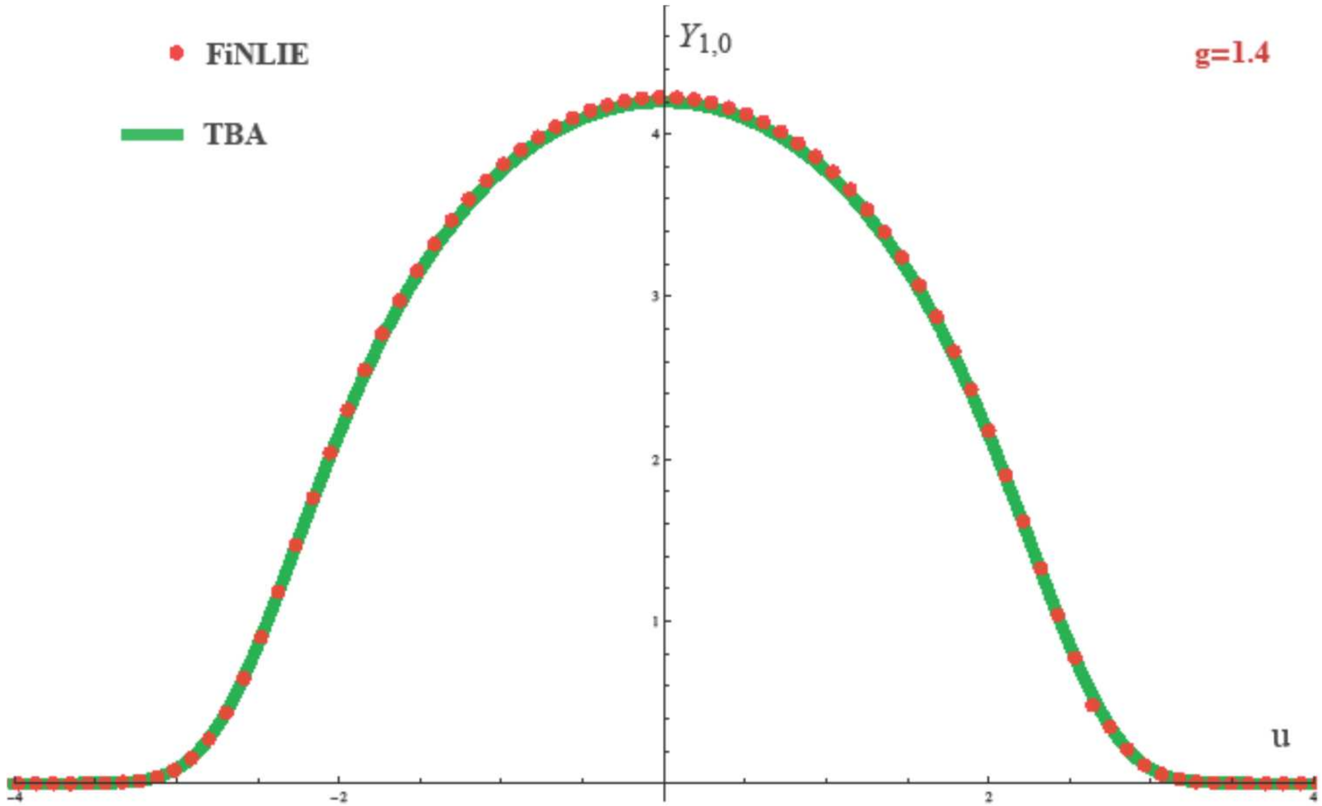
- Other functions can be excluded from the system of FiNLIE

- We manage to close the system of FiNLIE !
- Ready for analytic and numerical studies

Numerical solution of FiNLIE

- Writing our FiNLIE as $X = f(X), \qquad X = (\rho, \rho_2, \rho_3, \ldots)$

  we attempted to solve it on Mathematica by iterations: $X_{n+1} = f(X_n)$



- The coincidence with earlier results from the infinite Y-system (TBA) is very satisfactory!

Bethe roots and energy (anomalous dimension) of a state

- The Bethe roots characterizing a state are encoded into zeros of some q-functions (in particular q_{12}). Can be extracted from the absence of poles in T-functions in “physical” gauge. Or the old formula from TBA…

- The energy of a state can be extracted from the large u asymptotics

$$\log Y_{1,1} Y_{2,2} \simeq iP + \frac{iE}{u} + \mathcal{O}(u^{-2})$$

Conclusions

- Y-system obeys integrable Hirota dynamics – can be reduced to a finite system of non-linear integral eqs (FiNLIE).
- Our FiNLIE is based on a few natural, or even physical analyticity asumptions. Not using TBA but proven to have the same solution
- The form of our FiNLIE is perfectible
- Hirota dynamics provides a general method of solving quantum σ-models on a finite space circle.
- Possible "mathematical" subject: "sigma model"-like solutions of Hirota equations and associated Riemann-Hilbert problems for vasrious (a,s) boundary conditions.

END