On a new c-theorem involving branch point twist fields.

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This talk is based on the following papers:

- O. A. Castro-Alvaredo and E. L.,
  Higher particle form factors of branch point twist fields in integrable quantum field theories
  J. Phys. A **44**, 255401 (2011)
- O. A. Castro-Alvaredo, B. Doyon and E. L.,
  A new c-theorem involving branch point twist fields
  preprint: arXiv:1107.4280v1 [hep-th]

Overview

1. Introduction
   - branch point twist fields
   - Δ-sum rule and c-theorem
2. Negativity of $\dot{\Delta}_{\mathcal{T}}(r)$
   - near the IR fixed point
   - near the UV point
   - arguments for general distances.

Twist fields

(J.L. Cardy, O.A. Castro-Alvaredo, B. Doyon, 2008)
When considering n copies of a 2-dimensional QFT cyclically connected, we call branch point **twist fields** $\mathcal{T}$ the set of fields associated to the invariance under the permutation of two copies. Their correlation functions can be defined as

$$\langle \mathcal{T}(0)\mathcal{O}(x)\rangle \propto \int_{\mathcal{C}(0)} [d\varphi]\, e^{-S[\varphi]}\mathcal{O}(x)$$

If we are considering the CFT counterpart of $\mathcal{T}$ is a primary field, and its conformal weight can be evaluated

$$\Delta_{\mathcal{T}} = \bar{\Delta}_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

where c is the conformal charge of the model, while n is the number of copies.

Δ-sum rule and c-theorem

(G. Delfino, P. Simonetti, J.L. Cardy; 1996)
As $\mathcal{T}$ is a primary field in the conformal theory the Δ-sum rule can be applied, and we can define a function $\Delta_{\mathcal{T}}(r)$ for which

$$\frac{d\Delta_{\mathcal{T}}(r)}{dr} = \frac{r\left(\langle\Theta(r)\mathcal{T}(0)\rangle_{(\mathbb{R}^2)_n} - \langle\Theta\rangle_{(\mathbb{R}^2)_n}\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}\right)}{2\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}}$$

with $\Delta(\infty) = 0$ and $\Delta(0) = \Delta_{\mathcal{T}}$.
On the other hand we know that c-theorem holds
(A.B. Zamolodchikov; 1986)

$$\frac{dc(r)}{dr} = -\frac{3r^3}{2}\left(\langle\Theta(r)\Theta(0)\rangle_{(\mathbb{R}^2)_n} - \langle\Theta\rangle^2_{(\mathbb{R}^2)_n}\right)$$

with $c(\infty) = 0$.

The c-function has the well known properties:

- $c(r) \geq 0 \qquad \forall r \in \mathbb{R}^+$.
- $\dot{c}(r) \leq 0 \qquad \forall r \in \mathbb{R}^+$.
- $\dot{c}(r) = 0 \qquad r = r^*$, where $c(r^*) = c$.

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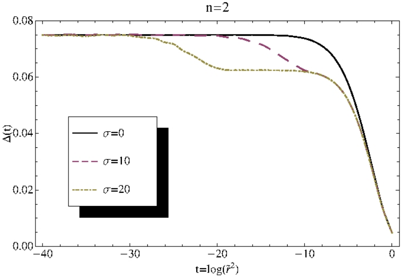
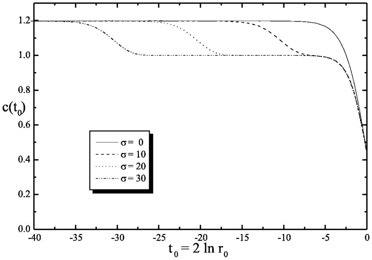
It is clear that

$$\Delta_{\mathcal{T}}(0) \propto c(0)$$

while we do not know if this holds out of the critical point. Suggestions that this might be possible come from the study of $\Delta_{\mathcal{T}}(r)$ for models which present plateaus in their RG-flows.

A typical example is $SU(3)_2$ Homogeneous sine-Gordon model *(Miramontes et al.; 1997)*

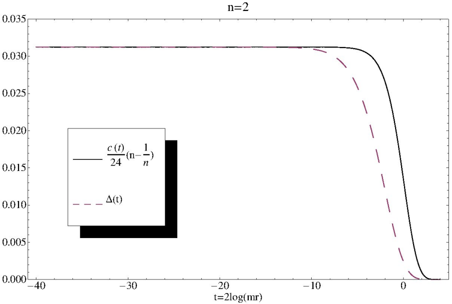




O.A. Castro-Alvaredo, A. Fring; 2000

in which $\Delta_{\mathcal{T}}(r)$ shows the same qualitative behavior of $c(r)$. From these pictures we could see that on every plateau the two functions are perfectly rescaled one on the other following the CFT rule.

Out of these plateaus though the two functions are consistently different, as we can see from the c and $\Delta_{\mathcal{T}}$ flow of the Ising model



$\Delta_{\mathcal{T}}(r)$ seems to satisfy all the properties of $c(r)$. On this consideration we base our analysis, and we try to demonstrate, or at least check these features for a general case.

Negativity of $\dot{\Delta}_{\mathcal{T}}(r)$

We begin noticing that

$$\dot{\Delta}_{\mathcal{T}}(r) \equiv \frac{r}{2}\left(\frac{\langle\Theta(r)\mathcal{T}(0)\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}} - \langle\Theta\rangle_{(\mathbb{R}^2)_n}\right) = \frac{n}{2}r\left(\langle\Theta(r)\rangle_{\mathcal{M}_0^n} - \langle\Theta\rangle_{\mathbb{R}^2}\right)$$

where $\mathcal{M}_0^n$ is the n-sheeted Riemann surface with a branch point at the origin and a branch cut on $\mathbb{R}^+$.
Hence what we are actually arguing is that

$$\langle\Theta(r)\rangle_{\mathcal{M}_0^n} - \langle\Theta\rangle_{\mathbb{R}^2} < 0 \qquad 0 < r < \infty$$

We divided our work in three different regions:

- near of IR point
- near the UV point
- general arguments in between.

Near the IR point

We know that in this region we can use form factors

$$F_k^{\mathcal{O}|a_1,a_2,\dots,a_k}(\theta_1,\theta_2,\dots,\theta_k) \equiv \langle 0|\mathcal{O}(0)|\theta_1,\theta_2,\dots,\theta_k\rangle_{in}^{a_1,a_2,\dots,a_k}$$

to create expansions for two point functions, which converge very fast.
We have

$$\langle\Theta(r)\rangle_{\mathcal{M}_0^n} - \langle\Theta\rangle_{\mathbb{R}^2} \simeq \sum_{a,b=1}^{n}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{2n(2\pi)^2}F_2^{\Theta|ab}(\theta_1,\theta_2)\left(\frac{F_2^{\mathcal{T}|ab}(\theta_1,\theta_2)}{\langle\mathcal{T}\rangle}\right)^* e^{-rm(\cosh\theta_1+\cosh\theta_2)}$$

If the massive theory is integrable then we know the form of these form factors and we have

$$\langle\Theta(r)\rangle_{\mathcal{M}_0^n} - \langle\Theta\rangle_{\mathbb{R}^2} = -\frac{m^2\sin\frac{\pi}{n}}{2n\pi}\int_{-\infty}^{\infty}dx\,\frac{K_0(2mr\cosh\frac{x}{2})}{\cosh\frac{x}{n}-\cos\frac{\pi}{n}}\,\frac{F_{\min}^{11}(x,1)}{F_{\min}^{11}(i\pi,1)}\,\frac{F_{\min}^{11}(x,n)^*}{F_{\min}^{11}(i\pi,n)^*},$$

It turns out that for integrable models of the type we are considering we have an integral representation for $F_{\rm min}$

$$F^{11}_{\rm min}(x,n) = \exp \int_0^\infty \frac{dt\, f(t)}{t \sinh(nt)} \sin^2\left[\frac{it}{2}\left(n + \frac{ix}{\pi}\right)\right]$$

where f(t) is a real function which is directly related to the S-matrix. We can use it to write

$$F^{11}_{\rm min}(x,1)F^{11}_{\rm min}(x,n)^* = \exp \int_0^\infty \frac{dt}{2t} f(t) \left(\frac{1-\cos\frac{tx}{\pi}\cosh nt}{\sinh nt} + \frac{1-\cos\frac{tx}{\pi}\cosh t}{\sinh t}\right)$$

This establishes that the quantity $\langle \Theta(r) \rangle_{\mathcal{M}_0^n} - \langle \Theta \rangle_{\mathbb{R}^2}$ is negative in the IR point region.

Near the UV point

(A. B. Zamolodchikov; 1989)
In this region it is particularly convenient to look to our massive IQFT as a perturbation of a CFT

$$S = S_{CFT}^{(n)} + g \sum_{j=1}^{n} \int_{\mathcal{M}_0^n} d^2 z_j \phi(z_j, \bar{z}_j)$$

Under this assumption we can express the trace of the energy momentum tensor as

$$\Theta(z, \bar{z}) = 4\pi g(1-\Delta)\phi(z, \bar{z})$$

so that we can switch our attention to the two point function of the perturbing field ϕ with the twist field $\mathcal{T}$.

We focus then on $\langle\phi(r)\rangle_{\mathcal{M}_0^n}$ rather than $\langle\Theta(r)\rangle_{\mathcal{M}_0^n}$ and write

$$\frac{\langle\phi(r)\mathcal{T}(0)\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \sum_{\mu=0}^{\infty} C_{\phi\mathcal{T}}^{\mu}(r)\frac{\langle\mathcal{O}_\mu\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}}$$

The functions $C_{\phi\mathcal{T}}^{\mu}(r)$ can be evaluated perturbatively, and we will consider the zeroth order.
We denote the leading term of the OPE $\mathcal{O}_0$ as $:\phi\mathcal{T}:$ so that

$$\frac{\langle\phi(r)\mathcal{T}(0)\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \tilde{C}_{\phi\mathcal{T}}^{:\phi\mathcal{T}:}r^{2(\Delta_{:\phi\mathcal{T}:}-\Delta-\Delta_{\mathcal{T}})}\frac{\langle:\phi\mathcal{T}:\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}} + \cdots$$

where $\tilde{C}_{\phi\mathcal{T}}^{:\phi\mathcal{T}:}$ is a dimensionless constant.

We can set the values of $\tilde{C}_{\phi\mathcal{T}}^{:\phi\mathcal{T}:}$ and $\Delta_{:\phi\mathcal{T}:}$ with considering the conformal transformation which connects the manifold $\mathcal{M}_0$ with $\mathbb{R}^2$

$$\langle\phi(z,\bar{z})\mathcal{O}(x,\bar{x})\rangle_{\mathcal{M}_0^n} = \frac{r^{2\Delta(\frac{1}{n}-1)}}{n^{2\Delta}}\langle\phi(0)(f*\mathcal{O})(f(x),f(\bar{x}))\rangle_{\mathbb{R}^2} + \dots$$

where $r=\sqrt{z\bar{z}}$.
We can now extract

$$\Delta_{:\mathcal{T}\phi:} = \frac{\Delta}{n} + \Delta_{\mathcal{T}} \quad \text{and} \quad \tilde{C}_{\phi\mathcal{T}}^{:\phi\mathcal{T}:} = \frac{1}{n^{2\Delta}}$$

$$\langle\Theta(r)\rangle_{\mathcal{M}_0^n} - \langle\Theta\rangle_{\mathbb{R}^2} = m^2\left(\frac{\alpha\beta(mr)^{2\Delta(\frac{1}{n}-1)}}{n^{2\Delta}} - \mu\right) + \cdots$$

where $4\pi g(1-\Delta) = \alpha m^{2-2\Delta}$, $\langle\Theta\rangle_{\mathbb{R}^2} = \mu m^2$ and

$$\frac{\langle :\phi\mathcal{T}:\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \beta m^{\frac{2\Delta}{n}},$$

and α, β and μ are all dimensionless constants.

- α is positive for theories we are looking at
- μ is expected to be positive for unitary models

For the Ising model

$$\langle\Theta(r)\rangle_{\mathcal{M}_0^n} - \langle\Theta\rangle_{\mathbb{R}^2} = -\frac{m^2}{n\pi 2^{\frac{1}{n}}}\cos\frac{\pi}{2n}\,\Gamma\left(\frac{1}{2}-\frac{1}{2n}\right)^2 (mr)^{\frac{1}{n}-1} + \ldots$$

General arguments

We can find distance-independent arguments for the negativity of $\langle\Theta(r)\rangle_{\mathcal{M}_0^n}$ looking to its variation with respect to n.
We need an argument for

$$\frac{d}{dn}\left(\langle\Theta(r)\rangle_{\mathcal{M}_0^n}\right) < 0 \qquad \text{for} \qquad n > 1$$

This argument comes from angular quantisation.
In this picture we managed to express

$$\frac{d}{dn}\langle\phi(r,0)\rangle_{\mathcal{M}_0^n} = -2\pi\left(\langle J\phi(r,0)\rangle_{\mathcal{M}_0^n} - \langle J\rangle_{\mathcal{M}_0^n}\langle\phi(r,0)\rangle_{\mathcal{M}_0^n}\right)$$

where J is related to the Hamiltonian (rotations generator).
Since the system is rotation invariant we have

$$\langle[J,\phi(r,0)]\rangle_{\mathcal{M}_0^n} = 0$$

J is composed of the kinetic part (CFT) and the potential $g\phi$ (perturbation), and the energy should be positively correlated to its potential.

We carried out the Klein-Gordon model as an example, and we are interested in the operator $\Theta \propto :\varphi^2:$ for which we can prove

$$\frac{d\langle:\varphi(r,0)^2:\rangle_{\mathcal{M}_0^n}}{dn} = -4\int_{-\infty}^{\infty} d\nu\,\nu\frac{\sinh(\pi\nu)}{\sinh^2(\pi n\nu)}|K_{i\nu}(mr)|^2 < 0$$

Conclusions

We have provided evidence that the function

$$\tilde{c}(r) = \frac{24n\Delta_{\mathcal{T}}(r)}{n^2 - 1}$$

satisfies all the properties of Zamolodchikov's c-function.

- We used a form factor expansion to check large distances
- we used the OPE of Θ and $\mathcal{T}$ to investigate the short distance behavior
- we gave general arguments which hold for general r and for every unitary theory.
- we proved that $\Delta_{\mathcal{T}}(r)$ is monotonically decreasing for the Ising and Klein-Gordon models.

THANKS FOR YOUR ATTENTION

Any questions?