A relativistic relative of GS supertrings on $AdS_5 \times S^5$

The Semi-Symmetric Space sine-Gordon theory

J. Luis Miramontes

U. Santiago de Compostela/IGFAE, (Spain)

$^{(*)}$ in collaboration with **Ben Hoare** (London) and **Timothy J. Hollowood** (Swansea)

Based on `arXiv:1107.0628, 1104.2429, ...`

The SSSG and SSSSG theories

Pohlmeyer, Eichenherr, Forger, D'Auria, Regge, ...'76-81

☞ $\boxed{\text{Symmetric space sine-Gordon (SSSG) theories}}$

Tseytlin'03
Mikhailov'05

- Relativistic integrable theories in 1+1 dimensions that are classically equivalent, via the Pohlmeyer reduction, to the non-relativistic (gauged fixed) world-sheet theories of strings on symmetric space spacetimes like $\mathbb{R}_t \times S^n$, $\mathbb{R}_t \times \mathbb{C}P^n$, $AdS_n \times S^1_\vartheta$, AdS_n, ...

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Hofman-Maldacena'06
Chen-N.Dorey-Okamura'06
....

- Admit soliton solutions that, for $\mathbb{R}_t \times S^n$ or $\mathbb{R}_t \times \mathbb{C}P^n$, are the images of the string giant magnons

- To describe the world-sheet theory of superstrings with all the fermionic degrees of freedom one has to generalize the SSSG theories to the case where the symmetric space is replaced by a semi-symmetric space $\mathcal{F}/G$, with $\mathcal{F}$ a supergroup

Serganova'83
Zarembo'10

  - $\mapsto$ $\mathbb{Z}_4$ automorphism: $\sigma^4 = 1, \quad \sigma(G) = G$
  - $\mapsto$ Lie superalgebra decomposition: $\mathfrak{f} = \underbrace{\mathfrak{f}_0}_{\text{even}} \oplus \underbrace{\mathfrak{f}_1}_{\text{odd}} \oplus \underbrace{\mathfrak{f}_2}_{\text{even}} \oplus \underbrace{\mathfrak{f}_3}_{\text{odd}}$

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Metsaev-Tseytlin'98

- $AdS_5 \times S^5 \quad \longrightarrow \quad PSU(2,2|4) \big/ SO(4,1) \times SO(5)$

Arutyunov-Frolov'08
Stefanski'08

- $AdS_4 \times \mathbb{C}P^3 \quad \longrightarrow \quad OSp(6|4) \big/ SO(3,1) \times U(3)$

- $\cdots\cdots\cdots\cdots\cdots$

Grigoriev-Tseytlin'08
Mikhailov-SchaferNakemi'08

☞ The relativistic SSSSG theory for

$$\boxed{\mathcal{F}/G = \frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)}}$$

is classically equivalent, via a fermionic generalization of Pohlmeyer reduction, to the non-relativistic (gauge fixed) Green-Schwarz superstring world-sheet theory on $AdS_5 \times S^5$ [$\rightarrow$ A. Tseytlin's talk]

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Mikhailov'05
Schmidtt'11

  - ☹ They have different Hamiltonian structures

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Mikhailov'05
Schmidtt'11

☹ They have different Hamiltonian structures

Grigoriev-Tseytlin'08

★ ...but the equivalence has been conjectured to remain in the (conformally invariant) quantum theory

☺ Poisson brackets are coordinated $\longrightarrow$ interpolating family of PBs

☞ The equivalence has already passed several tests:

Roiban-Tseytlin'09

- The $AdS_5 \times S^5$ SSSSG theory is UV-finite

Hoare-Iwashita-Tseytlin'09
Iwashita'10

- Semiclassical partition function matches with string theory at one loop

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Our approach:

- Focus not so much on the Lagrangian and perturbation theory but rather on the solitons: perturbative fields re-appear!
- Quantize the moduli space dynamics of the solitons yielding the semi-classical spectrum
- Conjecture the S-matrix by imposing all the axioms of S-matrix theory and solving the bootstrap (account for all poles on the physical strip)

The (relativistic) SSSSG Lagrangian

Grigoriev-Tseytlin'08
Mikhailov-SchaferNakemi'08

$$\boxed{\begin{aligned}\mathscr{L} = \mathscr{L}_{\mathrm{gWZW}}[G/H] - \frac{k}{\pi}\,\mathrm{STr}\left(\Lambda\gamma^{-1}\Lambda\gamma\right) \\ + \frac{k}{2\pi}\,\mathrm{STr}\left(\psi_+[\Lambda, D_-\psi_+] - \psi_-[\Lambda, D_+\psi_-] - 2\psi_+\gamma^{-1}\psi_-\gamma\right)\end{aligned}}$$

- $\mathfrak{psu}(2,2|4) = \underbrace{\mathfrak{f}_0}_{\text{even}} \oplus \underbrace{\mathfrak{f}_1}_{\text{odd}} \oplus \underbrace{\mathfrak{f}_2}_{\text{even}} \oplus \underbrace{\mathfrak{f}_3}_{\text{odd}}$
- $\gamma \in G = e^{\mathfrak{f}_0}$ and $A_\pm \longrightarrow$ bosonic fields
  $\psi_+ \in \mathfrak{f}_1,\ \psi_- \in \mathfrak{f}_3 \longrightarrow$ fermionic fields
- The potential is fixed by $\Lambda \equiv \mu\Lambda \in \mathfrak{f}_2$ (constant)

☞ Gauge symmetry group $\boxed{H = SU(2)^{\times 4} \subset G}$ $\quad [H, \Lambda] = 0$

☞ The coupling constant is the level of the WZW term $\equiv k$

Equations of motion

- Zero-curvature conditions $\boxed{\big[\mathcal{L}_\mu(z), \mathcal{L}_\nu(z)\big] = 0}$

$$\mathcal{L}_+(z) = \partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+\gamma + z\psi_+ - z^2\Lambda\,,$$
$$\mathcal{L}_-(z) = \partial_- + A_- + z^{-1}\gamma^{-1}\psi_-\gamma - z^{-2}\gamma^{-1}\Lambda\gamma$$

$\boxed{z = \text{spectral parameter}} \Rightarrow \boxed{\text{Classical integrability}}$

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☞ Integrability is controlled by the twisted affine loop superalgebra

$$\mathcal{L}\big(\mathfrak{psu}(2,2|4),\sigma\big) = \bigoplus_{n\in\mathbb{Z}}\bigoplus_{j=0}^{3} z^{4n+j}\otimes f_j = \bigoplus_{k\in\mathbb{Z}}\hat{\mathfrak{f}}_k, \qquad [\hat{\mathfrak{f}}_k, \hat{\mathfrak{f}}_l] \subset \hat{\mathfrak{f}}_{k+l}$$

Hidden symmetries

- Infinite (classical) symmetry algebra

$$\hat{\mathfrak{f}}^{\perp} = \mathrm{Ker}\big(\mathrm{ad}\Lambda\big) \cap \mathcal{L}\big(\mathfrak{psu}(2,2|4),\sigma\big) = \bigoplus_{k\in\mathbb{Z}} \hat{\mathfrak{f}}^{\perp}_{k}$$

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Hollowood-JLM'11
Hoare-Tseytlin'11

☞ The elements of grade ± 1 (or Lorentz spin $\pm 1/2$) generate SUSY transformations whose closure is the (exotic) $\mathcal{N} = (8|8)$ superalgebra

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- $\mapsto$ $\mathfrak{s}_{\pm 2}$: central elements corresponding to the components of p_μ
- $\mapsto$ $\mathfrak{s}_{\pm 1}$: generators of SUSY transformations
- $\mapsto$ $\mathfrak{s}_{0}$: generators of global gauge transformations $\equiv$ non-abelian R-symmetry group $SU(2)^{\times 4}$

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☞ $\mathfrak{s}$ is a finite subalgebra of $\boxed{\mathcal{L}\big(\mathfrak{p}(\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)),\sigma\big) \subset \hat{\mathfrak{f}}^{\perp}}$ and the derivation $\boxed{zd/dz}$ is the generator of Lorentz boosts

Goykhman-Ivanov'11
Hollowood-JLM'11

☞ The SUSY transformations act in a mildly non-local way on the Lagrangian fields

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$\Rightarrow$ Conjecture: the symmetry algebra becomes q-deformed in the quantum SSSSG theory $\boxed{U_q\big(\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^2\big)}$ $\boxed{q = e^{i\pi/k}}$

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☣ Different from the symmetry algebra of the superstring S-matrix

$$\boxed{\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3}$$

Solitons

Hollowood-JLM'11

- Continuous spectrum of relativistic non-abelian Q-ball kinks with bosonic and fermionic degrees of freedom

  labelled by $\varphi \in (0, \pi/2) \longrightarrow \boxed{m_\varphi = \frac{2k}{\pi}\mu \sin\varphi}$

  Non-trivial moduli space $\boxed{\mathfrak{M} = \frac{SU(2|2)}{U(2|1)} \times \frac{SU(2|2)}{U(2|1)}}$

- Grassmann coordinates arise from the non-compact AdS_5 sector

  $\longrightarrow$ turning off the Grassmann coordinates the solitons live in S^5

$\boxed{\text{Quantization of moduli space dynamics}}$

$$\downarrow$$

Semiclassical spectrum

Hilbert space of modules for the short (atypical) representations of

$\boxed{U_q(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2)}$ of dimension $\boxed{4a \times 4a}$

Mass spectrum $\boxed{m_a = \mu \sin\left(\frac{\pi a}{2k}\right)}$ $\boxed{a = 1, \ldots, k}$

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  but in the relativistic SSSSG theory the tower is truncated by k, the level of the WZW term.....

The relativistic S-matrix

- The SSSSG S-matrix with symmetry $U_q\big(\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^2\big)$ will be constructed as a graded tensor product of elementary blocks with symmetry $\boxed{U_q\big(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2\big)}$
- The blocks are obtained as a limit of the $U_q\big(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3\big)$ fundamental R-matrix of quantum-deformed Hubbard model

$\longrightarrow$ Beisert-Koroteev'08

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☞ $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2$ is a "finite" affine algebra

$$\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2 \subset \mathcal{L}\big(\mathfrak{su}(2|2), \sigma\big)$$

$\longrightarrow$ our relativistic S-matrix fits into the well known class of S-matrices associated to trigonometric solutions to the Yang-Baxter equation with affine quantum group $U_q(\hat{\mathfrak{g}})$ symmetry

Ahn-Bernard-LeClair'90
Hollowood '90
deVega-Fateev'91
. . . .

The extended $sl(2|2)$ superalgebra

- Generators

$$\text{Even} \to \mathfrak{sl}(2)^2 : \quad \mathfrak{R}^a{}_b\,, \quad \mathfrak{L}^\alpha{}_\beta$$

$$\text{Odd} \to \text{SUSY}: \quad \mathfrak{Q}^\alpha{}_b\,, \quad \mathfrak{S}^a{}_\beta \qquad \text{Centres:} \quad \boxed{\mathfrak{C}\,, \quad \mathfrak{P}\,, \quad \mathfrak{K}}$$

- $\boxed{\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3}$ algebra

$$[\mathfrak{R}^a{}_b, \mathfrak{R}^c{}_d] = \delta^c_b \mathfrak{R}^a{}_d - \delta^a_d \mathfrak{R}^c{}_b \qquad [\mathfrak{L}^\alpha{}_\beta, \mathfrak{L}^\gamma{}_\delta] = \delta^\gamma_\beta \mathfrak{L}^\alpha{}_\delta - \delta^\alpha_\delta \mathfrak{L}^\gamma{}_\beta$$

$$[\mathfrak{R}^a{}_b, \mathfrak{Q}^\gamma{}_d] = -\delta^a_d \mathfrak{Q}^\gamma{}_b + \tfrac{1}{2}\delta^a_b \mathfrak{Q}^\gamma{}_d \qquad [\mathfrak{L}^\alpha{}_\beta, \mathfrak{Q}^\gamma{}_d] = \delta^\gamma_\beta \mathfrak{Q}^\alpha{}_d - \tfrac{1}{2}\delta^\alpha_\beta \mathfrak{Q}^\gamma{}_d$$

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$$\boxed{\begin{aligned} &\{\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^c{}_\delta\} = \delta^c_b \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\delta \mathfrak{R}^c{}_b + \delta^c_b \delta^\alpha_\delta\, \mathfrak{C} \\ &\{\mathfrak{Q}^\alpha{}_b, \mathfrak{Q}^\gamma{}_d\} = \varepsilon^{\alpha\gamma}\varepsilon_{bd}\, \mathfrak{P} \qquad \{\mathfrak{S}^a{}_\beta, \mathfrak{S}^c{}_\delta\} = \varepsilon^{ac}\varepsilon_{\beta\delta}\, \mathfrak{K} \end{aligned}}$$

Quantum deformation $U_q(\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3)$

- Chevalley generators

$$\boxed{\begin{array}{c} [\mathfrak{E}_1, \mathfrak{F}_1] = [\mathfrak{H}_1]_q, \qquad \{\mathfrak{E}_2, \mathfrak{F}_2\} = -[\mathfrak{H}_2]_q, \qquad [\mathfrak{E}_3, \mathfrak{F}_3] = -[\mathfrak{H}_3]_q \\ q^{\mathfrak{H}_i}\mathfrak{E}_j = q^{A_{ij}}\mathfrak{E}_j q^{\mathfrak{H}_i}, \qquad q^{\mathfrak{H}_i}\mathfrak{F}_j = q^{-A_{ij}}\mathfrak{F}_j q^{\mathfrak{H}_i} \end{array}}$$

$+$ deformed Serre relations

- Cartan matrix: $A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ $\qquad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$

☞ $\boxed{q = e^{i\pi/k}}$

Defining representation(s)

☞ $U_q(\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3)$ has a family of 4-dimensional representations labelled by four parameters a, b, c, d constrained by

$$\boxed{(ad - qbc)(ad - q^{-1}bc) = 1}$$

- Centres

$$P = ab, \quad K = cd, \quad ad = [C + 1/2]_q\,, \quad bc = [C - 1/2]_q$$

$$\Rightarrow \boxed{[C]_q^2 - PK = [1/2]_q^2} \quad \equiv \quad (q\text{-deformed) shortening condition}$$

Beisert-Koroteev'08

- Beisert-Koroteev parameterization of $\{a, b, c, d\}$

$$a = \sqrt{g}\gamma, \qquad b = \frac{\sqrt{g}\alpha}{\gamma}\left(1 - q^{2C-1}\frac{x^+}{x^-}\right),$$

$$c = \frac{i\sqrt{g}\gamma}{\alpha}\frac{q^{-C+1/2}}{x^+}, \quad d = \frac{i\sqrt{g}}{\gamma} q^{C+1/2}\left(x^- - q^{-2C-1}x^+\right)$$

subject to

$$\frac{x^+}{q} + \frac{q}{x^+} - qx^- - \frac{1}{qx^-} + ig(q - q^{-1})\left(\frac{x^+}{qx^-} - \frac{qx^-}{x^+}\right) = \frac{i}{g}$$

☞ $\boxed{(g, q) \equiv \text{coupling constants}}$ $\boxed{(x^+, x^-) \equiv \text{dynamical variables}}$

$(\alpha, \gamma) \equiv$ normalization factors

The magnon and the soliton representations

- Magnon representation:

$$\boxed{q \longrightarrow 1} \quad (\text{or } k \longrightarrow \infty) \;\Rightarrow\; \boxed{\frac{x^+}{x^-} = e^{ip}}$$

$p \;\equiv\;$ world-sheet momentum $\qquad g \;\equiv\;$ string tension

$$[C]_q^2 - PK = [1/2]_q^2 \;\longrightarrow\; \epsilon(p) = \sqrt{1 + 4g^2 \sin^2(p/2)}$$

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Hoare-Hollowood-JLM'11

☞ Soliton representation:

$$\boxed{g \longrightarrow \infty} \;\Rightarrow\; \boxed{\frac{x^+}{x^-} = q, \quad C = 0}$$

$\Rightarrow \left[C\right]_q^2 - PK = \left[1/2\right]_q^2$ becomes a relativistic mass-shell condition

- $P = i\left[1/2\right]_q e^{-\theta} \equiv \frac{i}{\mu \sin(\pi/k)}\ p_-$

  $K = i\left[1/2\right]_q e^{+\theta} \equiv \frac{i}{\mu \sin(\pi/k)}\ p_+$

$$\boxed{-PK = \left[1/2\right]_q^2 \Rightarrow p_+ p_- = \mu^2 \sin^2(\pi/2k) = m_1^2}$$

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★ $U_q\left(\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3\right) \longrightarrow U_q\left(\mathfrak{psl}(2|2) \ltimes \mathbb{R}^2\right)$

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☞ Fundamental particle multiplet $V_1(\theta)$ of mass

$$m_1 = \mu \sin\left(\pi/2k\right) \longrightarrow \text{lightest semiclassical solitons}$$

$$m_1 \simeq \mu + O(1/k) \longrightarrow \text{perturbative modes}$$

Fundamental (relativistic) S-matrix

Hoare-Tseytlin'11
Hoare-Hollowood-JLM'11

- $\widetilde{S}_{11}(\theta_{12}) : V_1(\theta_1) \otimes V_1(\theta_2) \longrightarrow V_1(\theta_2) \otimes V_1(\theta_1)$ $\boxed{\theta_{12} = \theta_1 - \theta_2}$

$$\boxed{\widetilde{S}_{11}(\theta) = Y(\theta) Y(i\pi - \theta)\, \check{R}(e^{\theta})}$$

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Hoare-Tseytlin'11
Hoare-Hollowood-JLM'11

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★ Satisfies the Yang-Baxter equation

- $\check{R}$-matrix: basis $\{|\phi^a\rangle, |\psi^\alpha\rangle\}$, $x = e^\theta$, $q = e^{i\pi/k}$

$$\check{R}(x)\,|\phi^a\phi^a\rangle = A\,|\phi^a\phi^a\rangle\ , \qquad \check{R}(x)\,|\psi^\alpha\psi^\alpha\rangle = D\,|\psi^\alpha\psi^\alpha\rangle\ ,$$

$$\check{R}(x)\left|\phi^1\phi^2\right\rangle = \frac{q(A-B)}{q^2+1}\left|\phi^2\phi^1\right\rangle + \frac{q^2A+B}{q^2+1}\left|\phi^1\phi^2\right\rangle + \frac{C}{1+q^2}\left|\psi^1\psi^2\right\rangle - \frac{qC}{1+q^2}\left|\psi^2\psi^1\right\rangle\ ,$$

$$\check{R}(x)\left|\phi^2\phi^1\right\rangle = \frac{q(A-B)}{q^2+1}\left|\phi^1\phi^2\right\rangle + \frac{q^2B+A}{q^2+1}\left|\phi^2\phi^1\right\rangle - \frac{qC}{1+q^2}\left|\psi^1\psi^2\right\rangle + \frac{q^2C}{1+q^2}\left|\psi^2\psi^1\right\rangle\ ,$$

$$\check{R}(x)\left|\psi^1\psi^2\right\rangle = \frac{q(D-E)}{q^2+1}\left|\psi^2\psi^1\right\rangle + \frac{q^2D+E}{q^2+1}\left|\psi^1\psi^2\right\rangle + \frac{F}{1+q^2}\left|\phi^1\phi^2\right\rangle - \frac{qF}{1+q^2}\left|\phi^2\phi^1\right\rangle\ ,$$

$$\check{R}(x)\left|\psi^2\psi^1\right\rangle = \frac{q(D-E)}{q^2+1}\left|\psi^1\psi^2\right\rangle + \frac{q^2E+D}{q^2+1}\left|\psi^2\psi^1\right\rangle - \frac{qF}{1+q^2}\left|\phi^1\phi^2\right\rangle + \frac{q^2F}{1+q^2}\left|\phi^2\phi^1\right\rangle\ ,$$

$$\check{R}(x)\,|\phi^a\psi^\alpha\rangle = G\,|\psi^\alpha\phi^a\rangle + H\,|\phi^a\psi^\alpha\rangle\ , \qquad \check{R}(x)\,|\psi^\alpha\phi^a\rangle = K\,|\psi^\alpha\phi^a\rangle + L\,|\phi^a\psi^\alpha\rangle\ ,$$

$$A = \frac{(qx-1)(x+1)}{q^{1/2}x}\ , \qquad D = \frac{(q-x)(x+1)}{q^{1/2}x}\ ,$$

$$B = \frac{q^3-(q^3-2q^2+2q-1)x-x^2}{q^{3/2}x}\ , \qquad E = \frac{q^3x^2-(q^3-2q^2+2q-1)x-1}{q^{3/2}x}\ ,$$

$$C = F = \frac{i(q-1)(q^2+1)(x-1)}{q^{3/2}x^{1/2}}\ , \qquad G = L = x - x^{-1}\ ,$$

$$H = K = \frac{(q-1)(x+1)}{q^{1/2}x^{1/2}}\ .$$

- Y function

$$Y(\theta)Y(i\pi-\theta) = \frac{\mathcal{F}(\theta)}{2(q-q^{-1})}\ , \qquad \mathcal{F}(\theta) = \exp\left[-2\int_0^\infty \frac{dt}{t}\ \frac{\cosh^2(t(1-\frac{1}{k}))\sinh(t(1-\frac{\theta}{i\pi}))\sinh(\frac{t\theta}{i\pi})}{\sinh t\cosh^2 t}\right]$$

Bound states: the boostrap programme

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★ A non-trivial aspect of building a consistent QFT is to explain all the singularities of the S-matrix on the physical strip $0 < \operatorname{Im} \theta < \pi$ in terms of bound states and anomalous thresholds

☞ Bound states give rise to simple poles, and their positions have to mesh with the representation theory of the quantum group

If a bound state corresponding to a multiplet V_c is produced in the collision of V_a and V_b in the direct channel, then

1. $S_{ab}(\theta)$ has a simple pole at $\theta = iu_{ab}^c$
2. The representation $V_a(\theta_1) \otimes V_b(\theta_2)$ becomes reducible:
   $V_a \otimes V_b = V_c \oplus V_c^{\perp}$
3. $\boxed{\operatorname{Res} S_{ab}(iu_{ab}^c) : \; V_c^{\perp} \longrightarrow 0}$

☞ Res $S_{ab}(iu^c_{ab})$ is a weighted sum of "projectors"

$$V_c = \oplus_j V_c^{(j)}, \quad V_c^{(j)} = \text{irreducible representations of } U_q\big(\mathfrak{sl}(2)^{\times 2}\big)$$

$$S_{ab}(\theta) \sim \frac{i}{\theta - iu^c_{ab}} \sum_j \rho_j \, \mathbb{P}^{c,j}_{ab}, \qquad \mathbb{P}^{c,j}_{ab} = \mathcal{P}^{ba}_{c,j} \, \mathcal{P}^{c,j}_{ab}$$

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$$S_{ab}(\theta) \sim \frac{i}{\theta - iu_{ab}^c} \sum_j \rho_j \, \mathbb{P}_{ab}^{c,j}, \qquad \mathbb{P}_{ab}^{c,j} = \mathcal{P}_{c,j}^{ba} \, \mathcal{P}_{ab}^{c,j}$$

- ★ The scattering amplitudes of the bound state c with state d can be constructed in term of those of d with a and b.

$$\boxed{\begin{aligned} S_{dc}(\theta) = &\Big(\textstyle\sum_j \sqrt{|\rho_j|}\, \mathcal{P}_{ab}^{c,j} \otimes 1\Big) \Big(1 \otimes S_{db}(\theta + i\bar{u}_{b\bar{c}}^{\bar{a}})\Big) \\ &\times \Big(S_{da}(\theta - i\bar{u}_{a\bar{c}}^{\bar{b}}) \otimes 1\Big) \Big(1 \otimes \textstyle\sum_l \frac{1}{\sqrt{|\rho_l|}} \, \mathcal{P}_{c,l}^{ab}\Big) \end{aligned}}$$

Representation theory of $U_q(\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3)$

Zhang-Gould'05
Beisert'07
Beisert-Koroteev'08

☞ Long (typical) representations $\boxed{\{m, n\}} \mapsto \dim = 16(m+1)(n+1)$

Irreducible for generic values of C, P, K

- Become reducible but indecomposable for specific values of $[C]_q^2 - PK$ (shortening or BPS conditions)

$$V_{\{m,n\}} = V_{\text{sub-rep}} \oplus V^{\perp}, \qquad V_{factor} = V_{\{m,n\}} / V_{\text{sub-rep}}$$

$V_{\text{sub-rep}}, V_{factor} \equiv$ short (atypical) representations

☞ Short representation $\boxed{\langle m, n\rangle} \mapsto \dim = 4(m+1)(n+1)+4mn$

Exists for $\boxed{[C]_q^2 - PK = [(m+n+1)/2]_q^2} \longrightarrow$ Shortening condition

★ Fundamental representation $= \langle 0, 0\rangle$

Semiclassical solitons live in $\langle m, 0\rangle\,, \quad m \geq 0$

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★ Fundamental representation $= \langle 0, 0\rangle$

Semiclassical solitons live in $\langle m, 0\rangle\,, \quad m \geq 0$

- Tensor product $\langle m, 0\rangle \otimes \langle n, 0\rangle = \sum_{k=0}^{\min(m,n)} \{m+n-2k, 0\}$
- Multiplet splittings

$$\{m, 0\} \longrightarrow \langle m+1, 0\rangle \oplus \langle m, 1\rangle \ \text{ for } \ [C]_q^2 - PK = [(m+2)/2]_q^2$$

$$\longrightarrow \langle m-1, 0\rangle \oplus \langle m, 0\rangle_3 \ \text{ for } \ [C]_q^2 - PK = [m/2]_q^2$$

Bootstrap

☞ The shortening conditions indicate the location of the poles corresponding to the bound states

- $\widetilde{S}_{11}(\theta_{12}) : \overbrace{V_{\langle 0,0\rangle}(\theta_1) \otimes V_{\langle 0,0\rangle}(\theta_2)}^{V_{\{0,0\}}} \longrightarrow V_{\langle 0,0\rangle}(\theta_2) \otimes V_{\langle 0,0\rangle}(\theta_1)$

  $C_1 = C_2 = 0, \quad P = P_1 + P_2, \quad K = K_1 + K_2$

- Shortening condition for $\{0,0\} \to \langle 1,0\rangle \oplus \langle 0,1\rangle$

$$\boxed{-PK = [1]_q^2 \;\Rightarrow\; \theta_{12} = \pm\frac{i\pi}{k}}$$

☞ Bound state at $\theta = i\pi/k$ in the (factor) short representation $\langle 1,0\rangle$

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★ Consistent with the quantum group representation theory

$$\boxed{\mathrm{Res}\, \widetilde{S}_{11}(i\pi/k) : V_{\langle 0,1\rangle} \longrightarrow 0}$$

- $$V_{\{0,0\}} = \underbrace{V_{(2,0)} \oplus V_{(1,1)}^{(+)} \oplus V_{(0,0)}^{(+)}}_{\langle 1,0\rangle} \oplus \underbrace{V_{(0,0)}^{(-)} \oplus V_{(1,1)}^{(-)} \oplus V_{(0,2)}}_{\langle 0,1\rangle}$$

$$\operatorname{Res} \widetilde{S}_{11}\left(\tfrac{i\pi}{k}\right) \propto \frac{q+1}{2q^{1/2}}\mathbb{P}_{(2,0)} + \mathbb{P}_{(1,1)}^{(+)} + \frac{q^4 - q^3 + 4q^2 - q + 1}{2q^{3/2}(q+1)}\mathbb{P}_{(0,0)}^{(+)}$$

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★ $$\boxed{\widetilde{S}_{12}(\theta) = \left(1 \otimes \widetilde{S}_{11}(\theta + \tfrac{i\pi}{2k})\right)\left(\widetilde{S}_{11}(\theta - \tfrac{i\pi}{2k}) \otimes 1\right)\Big|_{V_{\langle 0,0\rangle}\otimes V_{\langle 1,0\rangle}}}$$

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☞ $\widetilde{S}_{12}(\theta)$ has four simple poles that can be explained in terms of the fusions $V_{\langle 0,0\rangle} \otimes V_{\langle 1,0\rangle} \to V_{\langle 2,0\rangle}$ and $V_{\langle 0,0\rangle} \otimes V_{\langle 1,0\rangle} \to V_{\langle 0,0\rangle}$ in the direct- and cross-channels

. . . .

☞ Quantum spectrum of particles associated to the representations

$\boxed{V_a(\theta) = \langle a-1, 0\rangle}$ with mass $\boxed{m_a = \mu \sin(\pi a/2k)}$

$\longrightarrow$ The semiclassical spectrum is exact!

Bound states of a and b correspond to simple poles at

$$\theta = i\pi(a+b)/2k \quad \text{and} \quad i\pi - i\pi|a-b|/2k$$

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★ This meshes with the quantum group representation theory

$$V_a(\theta_1) \otimes V_b(\theta_2) = \{a+b-2, 0\} \oplus \{a+b-4, 0\} \oplus \cdots \oplus \{|a-b|, 0\}$$

$\mapsto$ At $\boxed{\theta_{12} = i\pi(a+b)/2k}$ the representation $\{a+b-2, 0\}$ becomes reducible with a factor representation V_{a+b}, and $\widetilde{S}_{ab}(\theta_{12})$ is only non-vanishing on V_{a+b}

$\mapsto$ At $\boxed{\theta_{12} = i\pi - i\pi|a-b|/2k}$ the representation $\{|a-b|, 0\}$ becomes reducible with a factor representation $V_{|a-b|}$, and $\widetilde{S}_{ab}(\theta_{12})$ is only non-vanishing on $V_{|a-b|}$

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- Connection with semiclassical calculations

[⟶ Hoare-Iwashita-Roiban-Tseytlin'09-11]

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- Non-relativistic interpolating S-matrix

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☞ Quantum version of the interpolating classical Poisson brackets

★ Requires the construction of the interpolating dressing function

[⟶ Hoare-Hollowood-JLM in progress]