Coset Graphs in Logarithmic Minimal Models

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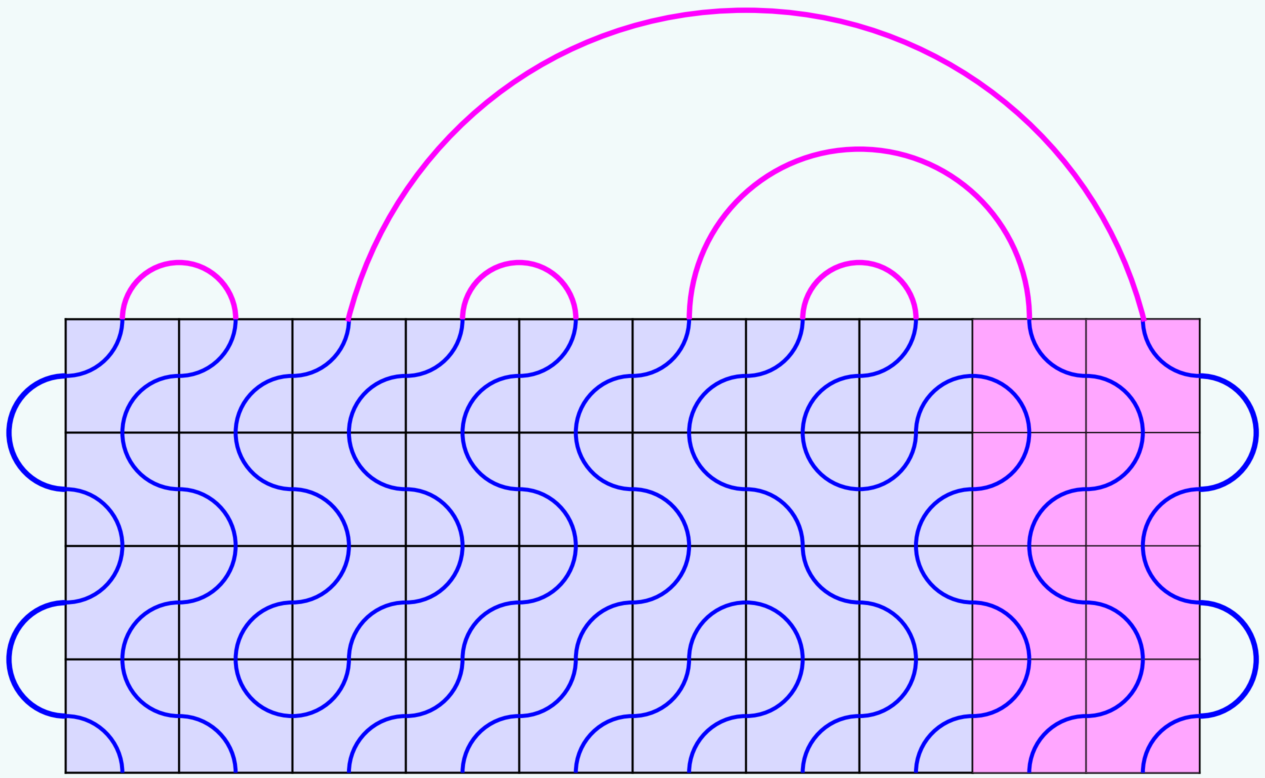

- PAP, J.Rasmussen, J.-B.Zuber, *Logarithmic minimal models*, J. Stat. Mech. P11017 (2006)
- PAP, J.Rasmussen, P.Ruelle, *Integrable boundary conditions and $\mathcal{W}$-extended fusion of the logarithmic minimal models $\mathcal{LM}(1,p)$*, J. Phys. A41 (2008)
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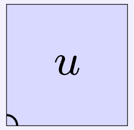
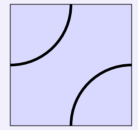
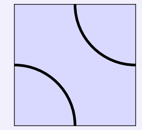
- J.Rasmussen, *$\mathcal{W}$-extended logarithmic minimal models*, Nucl. Phys. B807 (2008)
- PAP, J.Rasmussen, P.Ruelle, *Grothendieck ring and Verlinde-like formula for the $\mathcal{W}$-extended logarithmic minimal model $\mathcal{WLM}(1,p)$*, J. Phys. A43 (2010)
- PAP, J.Rasmussen, *Coset graphs in bulk and boundary logarithmic minimal models*, Nucl. Phys. B846 (2011)

Logarithmic Minimal Models $\mathcal{LM}(p,p')$

- **Face operators defined in planar Temperley-Lieb algebra (Jones 1999)**

$X(u) =$ 

$= \dfrac{\sin(\lambda-u)}{\sin\lambda}$  $+\ \dfrac{\sin u}{\sin\lambda}$  ; $X_j(u) = \dfrac{\sin(\lambda-u)}{\sin\lambda}\, I + \dfrac{\sin u}{\sin\lambda}\, e_j$

$1 \le p < p'$ coprime integers, $\lambda = \dfrac{(p'-p)\pi}{p'}$ = crossing parameter

u = spectral parameter, $\beta = 2\cos\lambda$ = fugacity of loops

$$Z = \sum_{\text{loop configs}} \left[\frac{\sin(\lambda-u)}{\sin\lambda}\right]^{N_1} \left[\frac{\sin u}{\sin\lambda}\right]^{N_2} \beta^{\#\,\text{loops}}$$

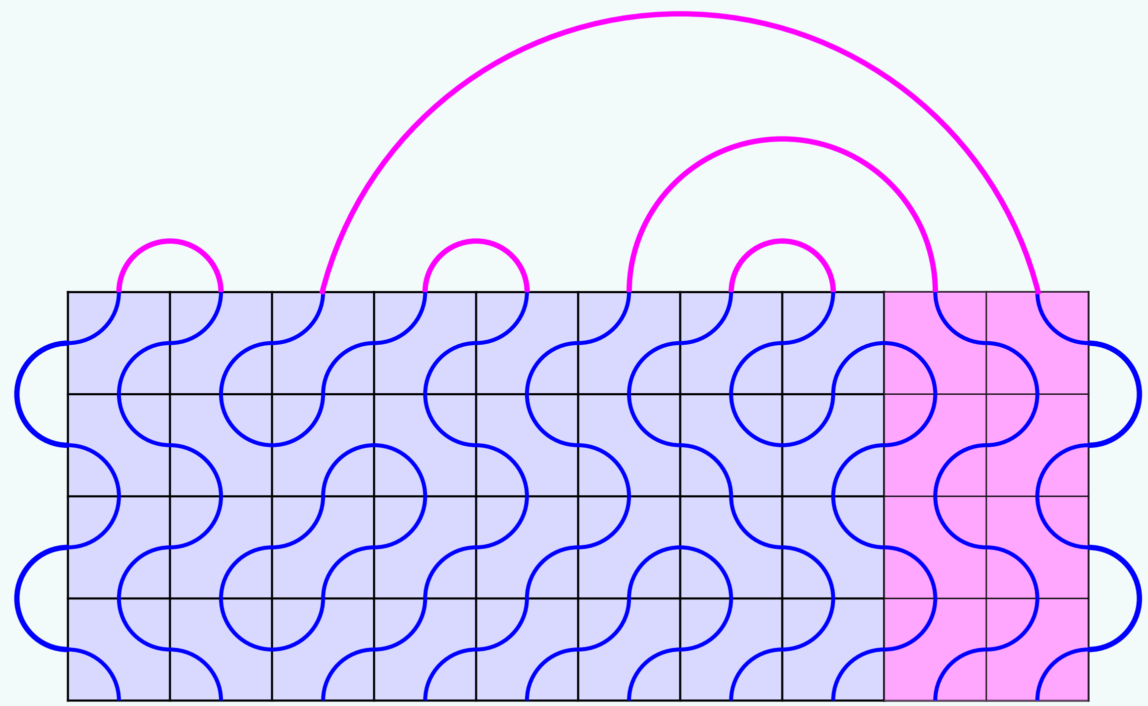
Planar Algebra
(Temperley-Lieb Algebra)

$\big\downarrow$ YBE

Nonlocal Statistical Mechanics
(Yang-Baxter Integrable Link Models)

continuum limit $\big\downarrow$ $\big\uparrow$ lattice realization

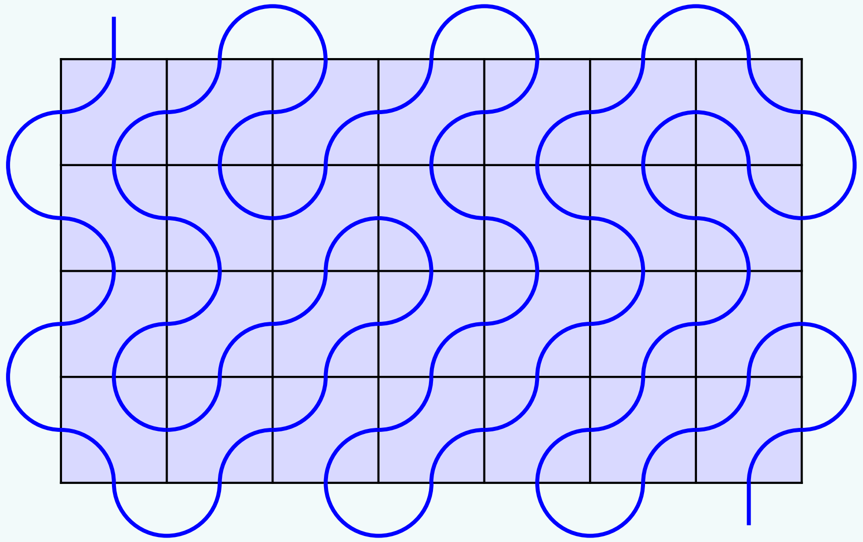
Logarithmic CFTs
(Logarithmic Minimal Models)



- Transfer matrix acts on space of link states
- No local degrees of freedom
- Only nonlocal degrees of freedom

Polymers and Percolation on the Lattice

- **Critical Dense Polymers:** $(p,p')=(1,2)$, $\lambda=\dfrac{\pi}{2}$

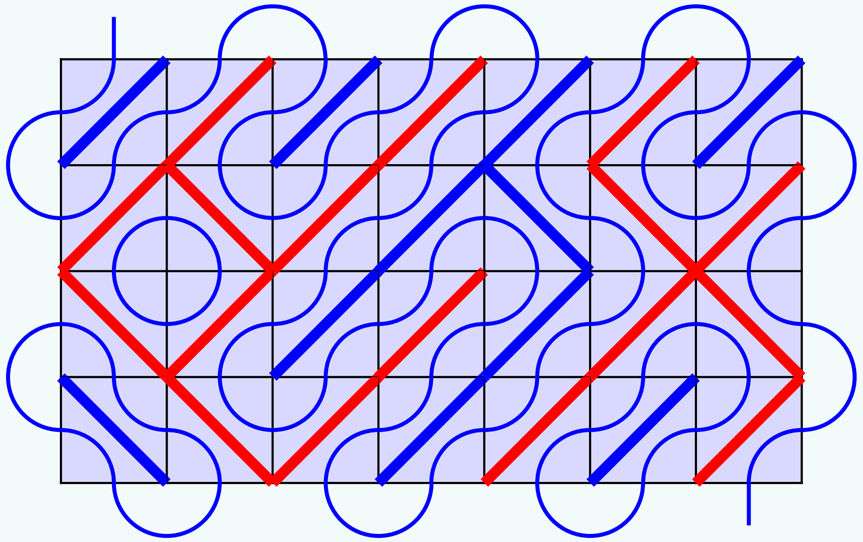


$$d_{path}^{\mathsf{SLE}}=2-2\Delta_{p,p'-1}=2,\qquad \kappa=\frac{4p'}{p}=8$$

$\Delta_{1,1}=0$ lies outside empty $\mathcal{M}(1,2)$ Kac table

$\beta=0 \;\Rightarrow\;$ no loops $\;\Rightarrow\;$ space filling dense polymer

- **Critical Percolation:** $(p,p')=(2,3)$, $\lambda=\dfrac{\pi}{3}$, $u=\dfrac{\lambda}{2}=\dfrac{\pi}{6}$ (isotropic)



bonds $=\mathcal{M}(2,3)$

loops $=\mathcal{LM}(2,3)$

1-1 mapping of config's

$$d_{path}^{\mathsf{SLE}}=2-2\Delta_{p,p'-1}=\frac{7}{4},\qquad \kappa=\frac{4p'}{p}=6$$

$\Delta_{2,2}=\frac{1}{8}$ lies outside rational $\mathcal{M}(2,3)$ Kac table

Bond percolation on the blue square lattice:

Critical probability $=p_c=\sin(\lambda-u)=\sin u=\frac{1}{2}$

$\beta=1\quad\Rightarrow\quad$ stochastic process

Representation Content

- The logarithmic minimal models have a very rich representation content:

Level	Reps
$\mathcal{W}$-Proj Grothendieck	$u(1)$ Char's $\varkappa_j(q)$
	$\uparrow$ $\uparrow$
$\mathcal{W}$-Proj (indec)	Proj Cover Proj Cover
	$\uparrow$ $\uparrow$
$\mathcal{W}$-Irred	Irred Min
Vir	Kac

- For reducible yet indecomposable representations, L_0 is non-diagonalizable with non-trivial Jordan blocks.

- A $\mathcal{W}$-projective representation is a 'maximal $\mathcal{W}$-indecomposable representation' (does not appear as a subfactor of any $\mathcal{W}$-indecomposable representation different from itself).
A projective cover of a $\mathcal{W}$-indecomposable representation $\hat{\mathcal{R}}$ is 'a $\mathcal{W}$-projective representation containing $\hat{\mathcal{R}}$ as a quotient'.

- Each $\mathcal{W}$-irreducible representation $\mathcal{W}(\Delta)$ admits a unique projective cover $\mathcal{P}(\Delta)$.

- The projective Grothendieck ring is the 'fusion ring of projective characters'. It is blind to indecomposibility. The generators $\mathcal{G}_{r,s}$ are equivalence classes of $\mathcal{W}$-projective reps that share a common character.

- These representations do not exhaust all representations!

Conformal Data of Logarithmic Minimal Models

- The logarithmic minimal models have central charges and conformal weights

$$c=1-\frac{6(p-p')^2}{pp'},\qquad \Delta_{r,s}=\frac{(p'r-ps)^2-(p-p')^2}{4pp'},\qquad r,s\ge 0,\quad 1\le p<p',\quad p,p'\ \text{coprime}$$

- They are nonunitary with effective central charges

$$\boxed{c^{\text{eff}}=c-24\Delta_{\text{min}}=1,\qquad \Delta_{\text{min}}=-\frac{(p-p')^2}{4pp'}}$$

- Characters: $(\hat{r}=0,1,\ldots 2p-1;\ \ \hat{s}=0,1,\ldots,p'-1;\ \ \ r=1,2,\ldots p-1;\ \ s=1,2,\ldots,p'-1)$

Kac: $$\chi^{\text{Vir}}_{r,s}(q)=q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)},\qquad r,s=1,2,\ldots,\infty$$

$\mathcal{W}$-Irreducible: $$\hat{\chi}_{\hat{r},\hat{s}}(q)=\frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k(k+\lfloor\tfrac{\hat{r}}{p}\rfloor)\Big(q^{(\hat{r}p'+\hat{s}p+2(k-1)pp')^2/4pp'}-q^{(\hat{r}p'-\hat{s}p+2kpp')^2/4pp'}\Big)$$

Minimal: $$\text{ch}_{r,s}(q)=\frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}\Big(q^{(rp'-sp+2kpp')^2/4pp'}-q^{(rp'+sp+2kpp')^2/4pp'}\Big)$$

$u(1)$: $$\varkappa^n_j(q)\ =\ \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}q^{(j+2kn)^2/4n},\qquad n=pp',\quad j=0,1,\ldots,2n$$

Central Questions:

- To what extent do $\mathcal{W}$-extended logarithmic minimal models $\mathcal{WLM}(p,p')$ resemble rational CFTs?
- Are the $\mathcal{W}$-extended logarithmic minimal models $\mathcal{WLM}(p,p')$ classified by graphs?

Kac Tables of Critical Percolation $\mathcal{LM}(2,3)/\mathcal{WLM}(2,3)$

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

$\hat{s}$					
2	$\frac{5}{8}$	2	$\frac{33}{8}$	7	
1	$\frac{1}{8}$	1	$\frac{21}{8}$	5	
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	
	0	1	2	3	$\hat{r}$

$\mathcal{W}$-Irreducible

$\hat{s}$					
2	$\frac{5}{8},\frac{5}{8}$	0,2 0,2	$\frac{1}{8},\frac{33}{8}$	0,1 2,7	
1	$\frac{1}{8},\frac{1}{8}$	0,1 0,1	$\frac{5}{8},\frac{21}{8}$	0,1 2,5	
0	$-\frac{1}{24}$	$\frac{1}{3},\frac{1}{3}$	$\frac{35}{24}$	$\frac{1}{3},\frac{10}{3}$	
	0	1	2	3	$\hat{r}$

Projective Covers

s				
3				
2		$\boxed{0}$		
1		$\boxed{0}$		
0				
	0	1	2	r

Minimal/Projective Covers

s				
3	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	
2	$\frac{5}{8}$	0	$\frac{1}{8}$	
1	$\frac{1}{8}$	0	$\frac{5}{8}$	
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	
	0	1	2	r

Proj Grothendieck
(Bordered Minimal Kac)

$$d_{r,s} = \begin{cases} \text{degree} \\ 2^{\text{rank}-1} \end{cases} = (2-\delta_{r,0}^{(p)})(2-\delta_{s,0}^{(p')}) = \begin{cases} 1, & \text{corner (mid blue)} \\ 2, & \text{edge (light blue)} \\ 4, & \text{interior (white)} \end{cases} \qquad \delta_{m,n}^{(N)} = \begin{cases} 1, & m = n \bmod N \\ 0, & \text{otherwise} \end{cases}$$

Projective Grothendieck Ring

- The projective Grothendieck ring is the ‘fusion ring of projective characters’. The generators $\mathcal{G}_{r,s}$ are equivalence classes of $\mathcal{W}$-projective reps that share a common character.
- The projective characters are naturally written in terms of $c=1$ $u(1)$ characters

$$\boxed{\chi[\mathcal{G}_{r,s}](q) = d_{r,s}\varkappa^n_{r,s}(q), \qquad \varkappa^n_{r,s}(q) = \tfrac{1}{2}\Big[\varkappa^n_{rp'-sp}(q) + \varkappa^n_{rp'+sp}(q)\Big], \qquad 0 \le r \le p; \quad 0 \le s \le p'}$$

with $n = pp'$ and

$$\varkappa^n_j(q) = \varkappa^n_{2n-j}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} q^{(j+2kn)^2/4n}, \qquad \Delta_j = \min\Big[\tfrac{j^2}{4n}, \tfrac{(2n-j)^2}{4n}\Big], \qquad j = 0, 1, \ldots, 2n$$

- The modular matrix $S = S^{p,p'}$ of the $\mathcal{W}$-projective characters gives rise to a Verlinde algebra which is the graph algebra of the twisted coset graph $A^{(2)}_{p,p'} = A^{(2)}_p \otimes A^{(2)}_{p'}/\mathbb{Z}_2$

$$\boxed{N_i N_j = \sum_{k=0}^{\frac{1}{2}(p+1)(p'+1)-1} {N_{ij}}^k N_k}$$

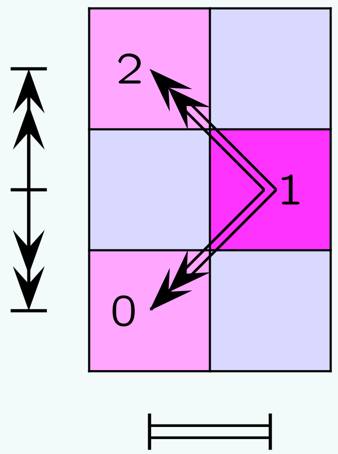
Here i, j, k run over allowed pairs (r, s) in the projective Grothendieck Kac table. The identity is $(r, s) = (0, 0)$ and structure constants are

$$ {N_{ij}}^k = {(N_i)_j}^k = \sum_{m=0}^{\frac{1}{2}(p+1)(p'+1)-1} \frac{S_{im}S_{jm}S_{mk}}{S_{0m}} \in \mathbb{N}_{\ge 0}$$

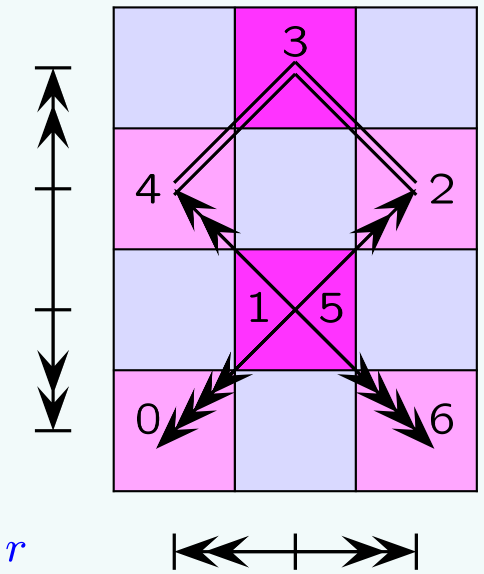
- The fusion rules of Grothendieck generators follow from the $\mathcal{W}$-projective fusion algebra (Rasmussen 2009). The modular matrix S ($S^2 = I, S \neq S^T$) diagonalizes these fusion rules!

Projective Grothendieck Kac Tables as Coset Graphs

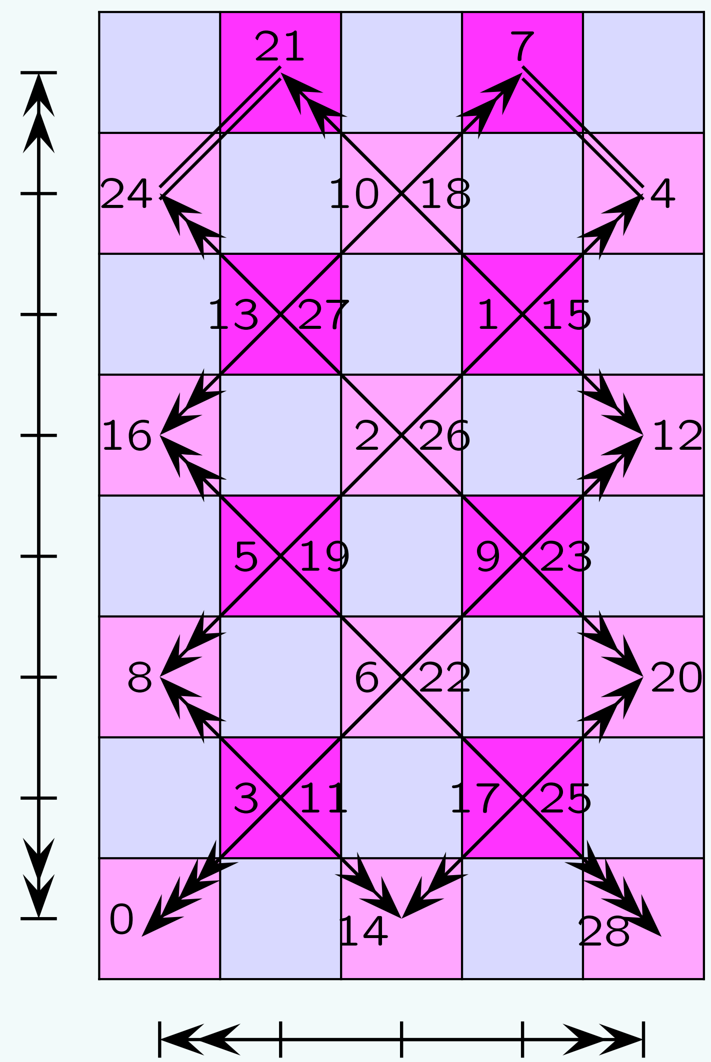
s \ r	0	1
2	$\frac{3}{8}$	$-\frac{1}{8}$
1	0	0
0	$-\frac{1}{8}$	$\frac{3}{8}$



s \ r	0	1	2
3	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
2	$\frac{5}{8}$	0	$\frac{1}{8}$
1	$\frac{1}{8}$	0	$\frac{5}{8}$
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$



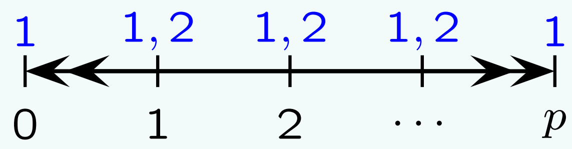
s \ r	0	1	2	3	4
7	$\frac{775}{112}$	$\frac{27}{7}$	$\frac{187}{112}$	$\frac{5}{14}$	$-\frac{9}{112}$
6	$\frac{81}{16}$	$\frac{5}{2}$	$\frac{13}{16}$	0	$\frac{1}{16}$
5	$\frac{391}{112}$	$\frac{10}{7}$	$\frac{27}{112}$	$-\frac{1}{14}$	$\frac{55}{112}$
4	$\frac{247}{112}$	$\frac{9}{14}$	$-\frac{5}{112}$	$\frac{1}{7}$	$\frac{135}{112}$
3	$\frac{135}{112}$	$\frac{1}{7}$	$-\frac{5}{112}$	$\frac{9}{14}$	$\frac{247}{112}$
2	$\frac{55}{112}$	$-\frac{1}{14}$	$\frac{27}{112}$	$\frac{10}{7}$	$\frac{391}{112}$
1	$\frac{1}{16}$	0	$\frac{13}{16}$	$\frac{5}{2}$	$\frac{81}{16}$
0	$-\frac{9}{112}$	$\frac{5}{14}$	$\frac{187}{112}$	$\frac{27}{7}$	$\frac{775}{112}$



The $A_p^{(2)}$ Fusion Algebra

- The *twisted* affine $A_p^{(2)}$ Dynkin graph is the 'classifying' graph for $\mathcal{WLM}(1,p)$.

$A_p^{(2)}$



$$N_1 = \begin{pmatrix} 0 & 1 & & & & \\ 2 & 0 & 1 & & & \\ & 1 & \cdot & & & \\ & & & \cdot & 1 & \\ & & & 1 & 0 & 2 \\ & & & & 1 & 0 \end{pmatrix} = \text{asymmetric adjacency}$$

The nodes $j = 0, 1, \ldots, p$ of the $A_p^{(2)}$ graph are associated with the characters $\varkappa_j^n(q)$.

- $C = 2I - N_1^T$ is a generalized (*symmetrizable*) Cartan matrix. So $D^{1/2} N_1^T D^{-1/2}$ is symmetric and $A = N_1^T$ is similar to a real symmetric matrix with real eigenvalues and eigenvectors

$$\boxed{D = \text{diag}(d_0, d_1, \ldots, d_p) = \text{diag}(1, 2, 2, \ldots, 2, 1)}$$

The entries of the left and right Perron-Frobenius eigenvectors are shown above in blue. The eigenvalues are

$$\lambda_r^p = 2 \cos \frac{r\pi}{p} = \text{eigvals}, \qquad r = 0, 1, \ldots, p$$

- More generally, let $\boldsymbol{a}$, $\check{\boldsymbol{a}}$ be the right/left Perron-Frobenius eigenvectors of an A-D-E twisted affine graph, with entries a_r and $\check{a}_r$ given by the Coxeter/dual Coxeter labels. Then

$$d_r = d_r^{(p)} = \frac{\check{a}_0}{a_0} \frac{a_r}{\check{a}_r} \in \mathbb{N}, \qquad r = 0, 1, 2, \ldots \quad \text{with} \quad d_0 = 1$$

- In particular, for $A_p^{(2)}$

$$\boldsymbol{a} = (1, 2, 2, \ldots, 2, 1), \qquad \check{\boldsymbol{a}} = (1, 1, 1, \ldots, 1, 1)$$

$$\boxed{d_r = d_r^{(p)} = a_r = \text{Coxeter labels}}$$

$\mathcal{WLM}(p,p')$ Twisted Affine Coset Graphs

- The 'classifying' graph for $\mathcal{WLM}(p,p')$ is the coset graph

$$\boxed{A^{(2)}_{p,p'} = A^{(2)}_{p} \otimes A^{(2)}_{p'} / \mathbb{Z}_2}$$

with $\frac{1}{2}(p+1)(p'+1)$ nodes. The $\mathbb{Z}_2$ quotient is taken with respect to the $\mathbb{Z}_2$ Kac-table symmetry. The eigenvalues of the coset graph adjacency matrices are

$$\lambda^{p,p'}_{r,s} = \lambda^{p,p'}_{p-r,p'-s} = 4\cos\frac{r\pi}{p}\cos\frac{s\pi}{p'}, \qquad r = 0,1,\ldots,p; \quad s = 0,1,\ldots,p'$$

- The coset graphs $A^{(2)}_{p,p'}$ are symmetrizable, with degrees

$$\boxed{d_{r,s} = d^{(p)}_r d^{(p')}_s = \begin{cases} 1, & (r,s) \text{ is a corner} \\ 2, & (r,s) \text{ is on an edge} \\ 4, & (r,s) \text{ is in the interior} \end{cases} \qquad D = \mathrm{diag}(\ldots, d_{r,s}, \ldots)}$$

The ranks of projective covers are thus related (through the Coxeter labels) to data of twisted affine Dynkin graphs. The diagonal matrix D for critical percolation $\mathcal{WLM}(2,3)$ is

$$D = \mathrm{diag}(d_{0,0}, d_{1,1}, d_{2,2}, d_{1,3}, d_{0,2}, d_{2,0}) = \mathrm{diag}(1,4,2,2,2,1) = \text{graph valencies}$$

- The $A^{(2)}_{p,p'}$ graph algebra, generated by the fundamentals $N_{1,0}$ and $N_{0,1}$, is

$$N_{r,s}N_{r',s'} = \sum_{r'',s''} N_{rs,r's'}{}^{r''s''} N_{r'',s''}$$

with structure constants

$$N_{rs,r's'}{}^{r''s''} = \frac{d_{r,s}d_{r',s'}}{4d_{r'',s''}}\Big(\delta_{r'',|r-r'|} + \delta_{r'',p-|p-r-r'|}\Big)\Big(\delta_{s'',|s-s'|} + \delta_{s'',p'-|p'-s-s'|}\Big) \in \mathbb{N}_0$$

Projective and Minimal A-Type Modular Invariants

Conjecture: The $\mathcal{WLM}(p,p')$ partition functions decompose into a sum of separate modular invariant sesquilinear forms in $\mathcal{W}$-projective and minimal characters

$$\boxed{Z = Z^{\text{Proj}} + Z^{\text{Min}}}$$

- The coset graphs provide new expressions for the diagonal A-type modular invariants in $\mathcal{W}$-projective characters considered by Feigin et al (2006) and Wood (2010)

$$\boxed{Z^{\text{Proj}}_{p,p'}(q) = \tfrac{1}{2}\sum_{r=0}^{p}\sum_{s=0}^{p'} d_{r,s}\,|\varkappa^n_{r,s}(q)|^2 = \tfrac{1}{2}\sum_{r=0}^{p}\sum_{s=0}^{p'} \frac{1}{d_{r,s}}\Big|\chi[\mathcal{G}_{r,s}](q)\Big|^2 = \sum_{\hat{r}=0}^{2p-1}\sum_{\hat{s}=0}^{p'-1}\frac{1}{d^2_{\hat{r},\hat{s}}}|\chi[\hat{\mathcal{P}}_{\hat{r},\hat{s}}](q)|^2}$$

The factors of $\frac{1}{2}$ in the first two double sums reflect the $\mathbb{Z}_2$ Kac-table symmetry, $|\varkappa_{0,0}(q)|^2$ appears with multiplicity 1 and all multiplicities are nonnegative integers.

- The coset graphs also encode the rational minimal A-type modular invariants

$$\boxed{Z^{\text{Min}}_{p,p'}(q) = \tfrac{1}{2}\sum_{r=1}^{p-1}\sum_{s=1}^{p'-1}|\text{ch}_{r,s}(q)|^2 = \tfrac{1}{2}\sum_{r=0}^{p}\sum_{s=0}^{p'}\Big|\varkappa^n_{rp'-sp}(q) - \varkappa^n_{rp'+sp}(q)\Big|^2}$$

where the factors of $\frac{1}{2}$ reflect a $\mathbb{Z}_2$ Kac-table symmetry.

A-Type $\mathcal{WLM}(p,p')$ Modular Invariants

- Assuming $Z^{\text{Proj}} \neq 0$ and that the operator with minimal conformal weight enters exactly once, the A-type $\mathcal{WLM}(p,p')$ modular invariant partition functions must be of the form

$$\boxed{Z_{p,p'}(q) = Z^{\text{Proj}}_{p,p'}(q) + n_{p,p'} Z^{\text{Min}}_{p,p'}(q), \qquad n_{p,p'} \in \mathbb{Z}}$$

For $p=1$, $Z^{\text{Min}}_{1,p'}(q) = 0$ and $Z_{p,p'}(q) = Z^{\text{Proj}}_{p,p'}(q)$.

Conjecture: For $p>1$, the physical A-type $\mathcal{WLM}(p,p')$ modular invariants are given by

$$\boxed{n_{p,p'} = 2}$$

- This generalizes a result of Gaberdiel, Runkel & Wood 2011 who recently constructed a consistent bulk theory of $\mathcal{WLM}(2,3)$.
- Explicitly, for symplectic fermions $\mathcal{WLM}(1,2)$ and critical percolation $\mathcal{WLM}(2,3)$, the modular invariant partition functions are

$$Z_{1,2}(q) = |\varkappa_{-\frac{1}{8}}(q)|^2 + 2|\varkappa_0(q)|^2 + |\varkappa_{\frac{3}{8}}(q)|^2$$

$$Z_{2,3}(q) = |\varkappa_{-\frac{1}{24}}(q)|^2 + \Big|\varkappa_0(q) + \varkappa_1(q)\Big|^2 + 2|\varkappa_{\frac{1}{8}}(q)|^2 + 2|\varkappa_{\frac{1}{3}}(q)|^2 + 2|\varkappa_{\frac{5}{8}}(q)|^2 + |\varkappa_{\frac{35}{24}}(q)|^2 + 2|\text{ch}_0(q)|^2$$

with $\text{ch}_0(q) = 1$.

Summary and Open Questions

- $\mathcal{W}$-projective representations emerge as *building blocks* of logarithmic minimal models playing the role of irreducible representations in rational CFTs. Both the boundary and bulk A-type logarithmic minimal models are 'classified' by twisted affine coset graphs.

- Multiplication in the Grothendieck ring associated to $\mathcal{W}$-projective representations leads to a *standard* Verlinde-like formula involving the twisted affine coset graphs. This yields compact formulas for the conformal partition functions with $\mathcal{W}$-projective boundary conditions.

- The conjectured torus modular invariant partition functions are naturally encoded by the twisted affine coset graphs.

- Is there an A-D-E classification of the logarithmic Verlinde fusion graphs à la Behrend, Pearce, Petkova and Zuber?

- Is there a corresponding A-D-E classification of the logarithmic modular invariant partition functions à la Cappelli, Itzykson and Zuber?

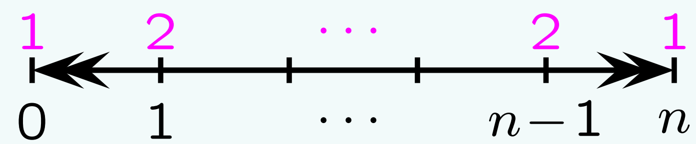
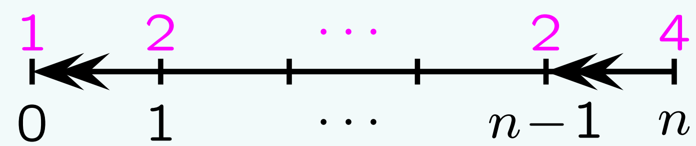
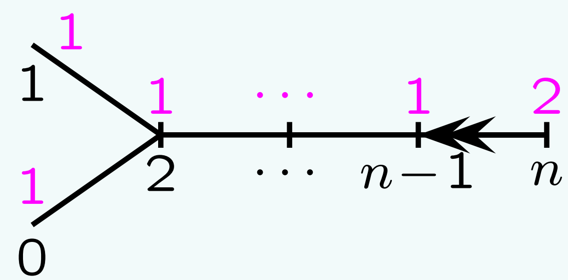
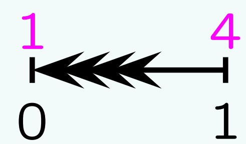
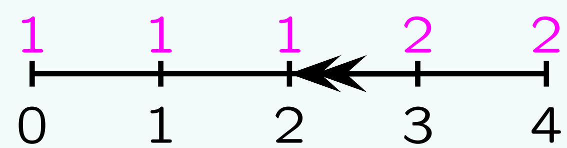
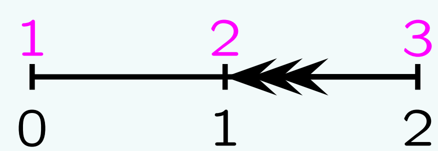
- Is there a logarithmic coset construction à la Goddard, Kent and Olive?

- Are there corresponding D and E logarithmic minimal models on the lattice?

- As yet, there is no lattice derivation of the conjectured torus modular invariant partition functions apart from $\mathcal{WLM}(1,2)$.

A-D-E Twisted Affine Lie Algebras

- Twisted affine Lie algebras associated with symmetrizable (generalized) Cartan matrices have been classified by Kac-Moody.

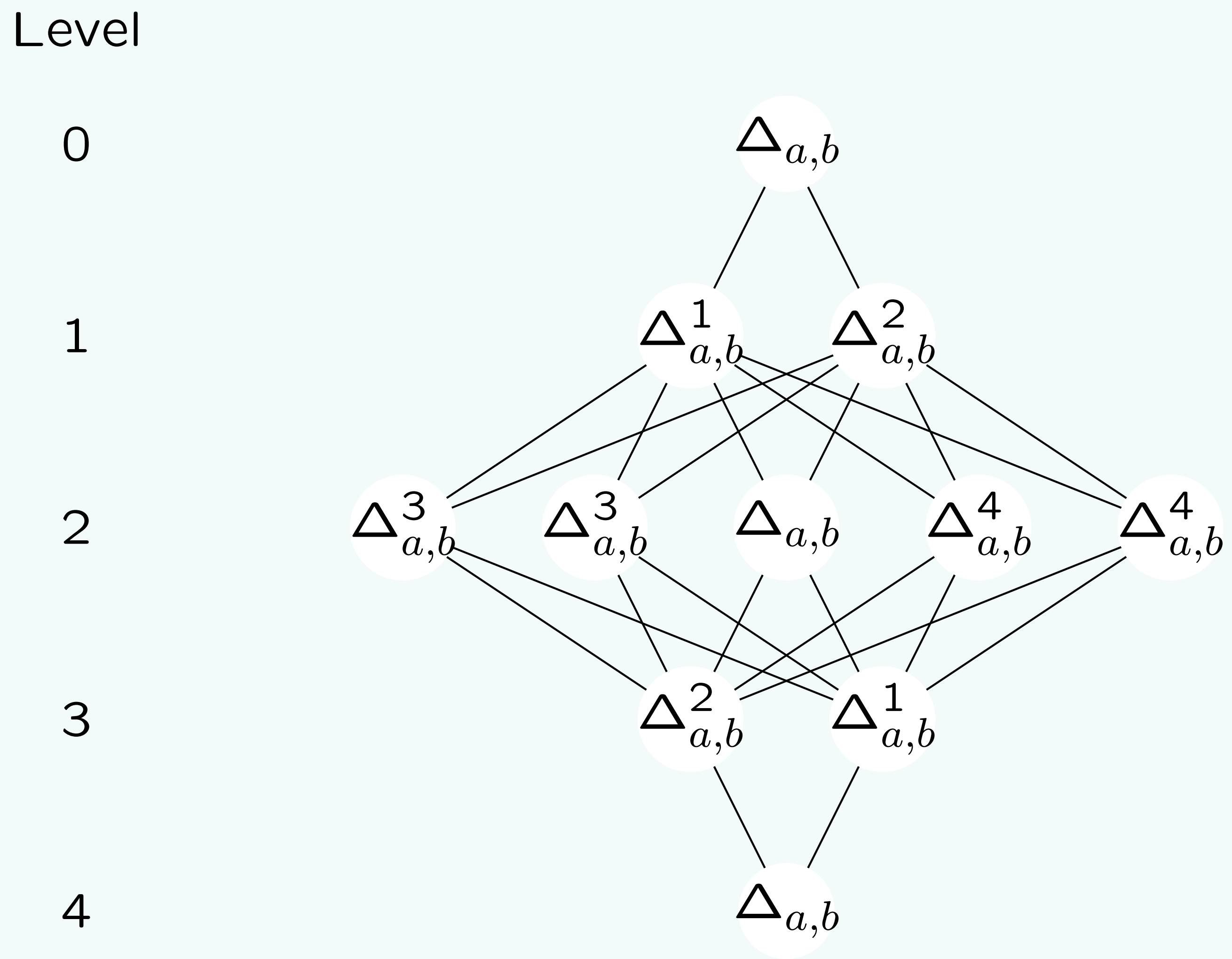
G	Kac	n	p	Dynkin Diagram d_k	Exponents $s \in$ Exp(G)
$A_n^{(2)}$	$D_{n+1}^{(2)}$	$n \geq 2$	n		$0, 1, 2, \ldots, n$
$\tilde{A}_n^{(2)}$	$A_{2n}^{(2)}$	$n \geq 2$	$2n$		$0, 2, 4, \ldots, 2n$
$D_n^{(2)}$	$A_{2n-1}^{(2)}$	$n \geq 3$	$2n-2$		$0, 2, 4, \ldots, 2n-2, n-1$
$\tilde{A}_1^{(2)}$	$A_2^{(2)}$	$n = 1$	2		$0, 2$
$E_4^{(2)}$	$E_6^{(2)}$	$n = 4$	12		$0, 4, 6, 8, 12$
$D_4^{(3)}$	$D_4^{(3)}$	$n = 1$	4		$0, 2, 4$

- The eigenvalues of the adjacency matrix of G are of the form

$$\boxed{2\cos\frac{s\pi}{p}, \qquad s \in \mathrm{Exp(G)}, \qquad |G| = \#\text{nodes of } G = n+1}$$

Minimal Projective Covers

- The conjectured structure of $\mathcal{P}_{a,b}$ for $1 \le a \le p-1$ and $1 \le b \le p'-1$ is



$$\Delta^1_{a,b} = \Delta_{p+a,p'-b}, \qquad \Delta^2_{a,b} = \Delta_{2p-a,b}, \qquad \Delta^3_{a,b} = \Delta_{3p-a,p'-b}, \qquad \Delta^4_{a,b} = \Delta_{2p+a,b}$$

- The corresponding characters are

$$\begin{aligned} \chi[\mathcal{P}_{a,b}](q) &= 2\,\mathrm{ch}_{a,b}(q) + \tfrac{1}{2}\chi[\hat{\mathcal{R}}^{a,b}_{p,p'}](q) \\ &= 3\,\mathrm{ch}_{a,b}(q) + 2\Big[\chi_{p+a,p'-b}(q) + \chi_{2p-a,b}(q) + \chi_{3p-a,p'-b}(q) + \chi_{2p+a,b}(q)\Big] \end{aligned}$$