Boundary Conditions and Kac Representations in Logarithmic Minimal Models

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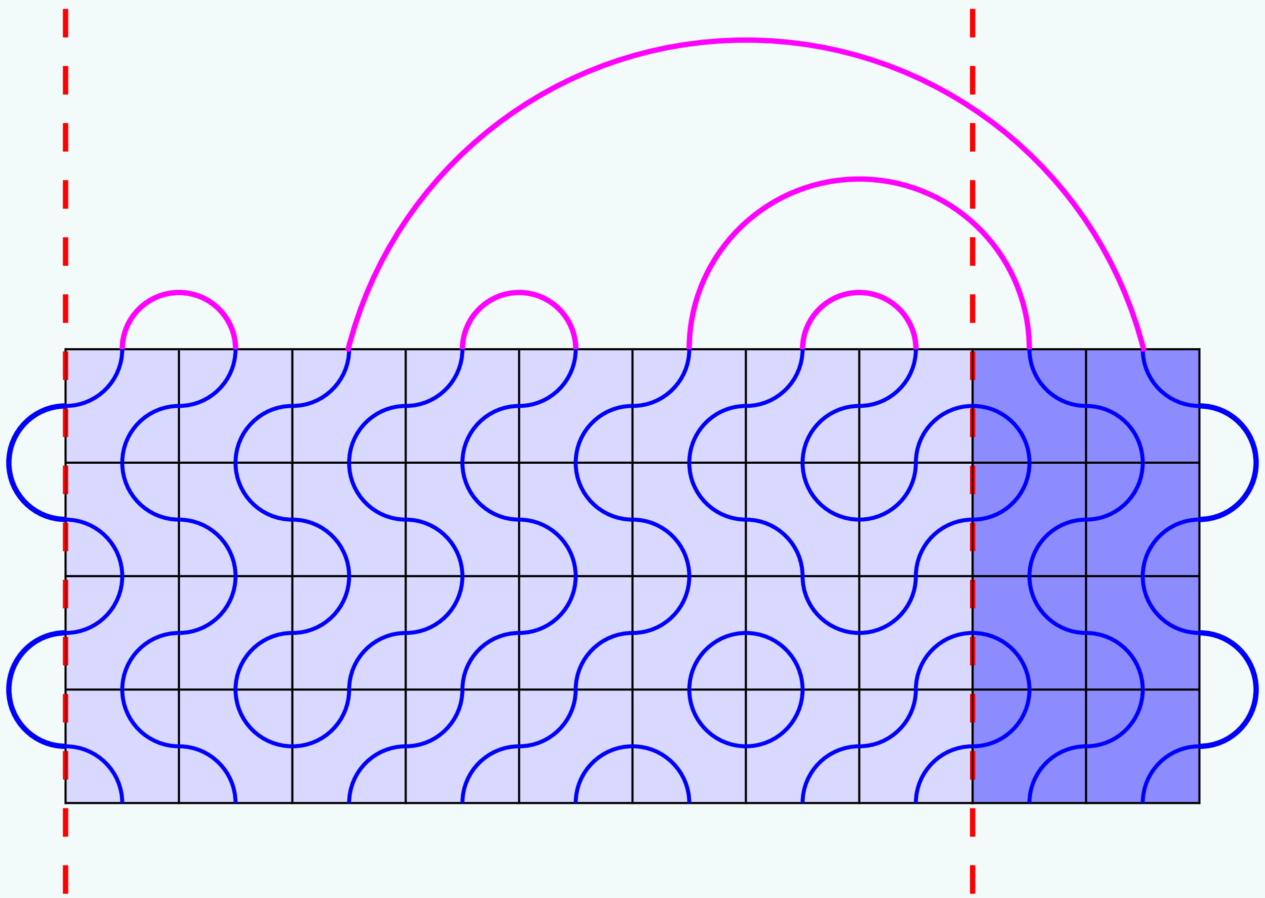
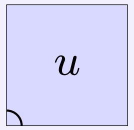
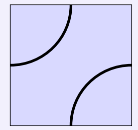
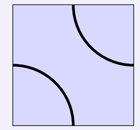


Logarithmic Minimal Models $\mathcal{LM}(p,p')$

Face operators defined in planar Temperley-Lieb algebra (Jones 1999)

$X(u) \;=$  $= \dfrac{\sin(\lambda-u)}{\sin\lambda}$  $+ \dfrac{\sin u}{\sin\lambda}$  ; $\qquad X_j(u) \;=\; \dfrac{\sin(\lambda-u)}{\sin\lambda}\, I + \dfrac{\sin u}{\sin\lambda}\, e_j$

$1 \le p < p'$ coprime integers, $\qquad \lambda \;=\; \dfrac{(p'-p)\pi}{p'} \;=$ crossing parameter

$u \;=$ spectral parameter, $\qquad \beta \;=\; 2\cos\lambda \;=$ fugacity of loops

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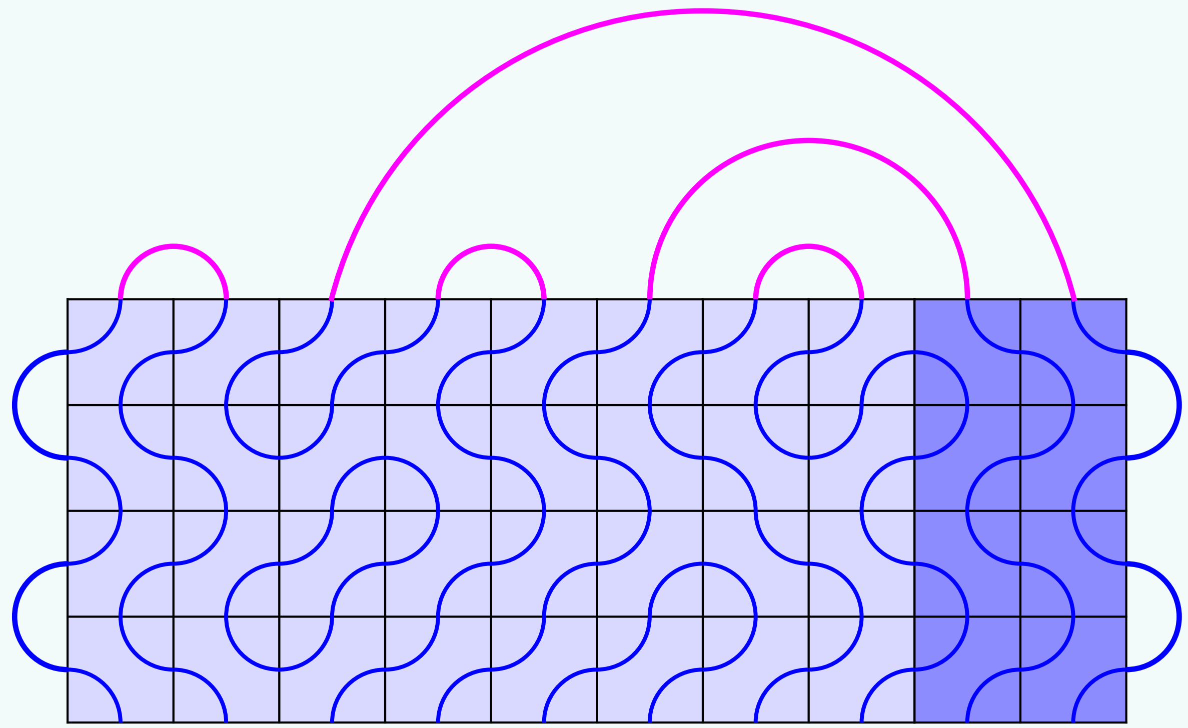
Planar Algebra
(Temperley-Lieb Algebra)

↓ YBE

Non-Local Statistical Mechanics
(Yang-Baxter Integrable Link Models)

↓ continuum limit ↑ lattice realization

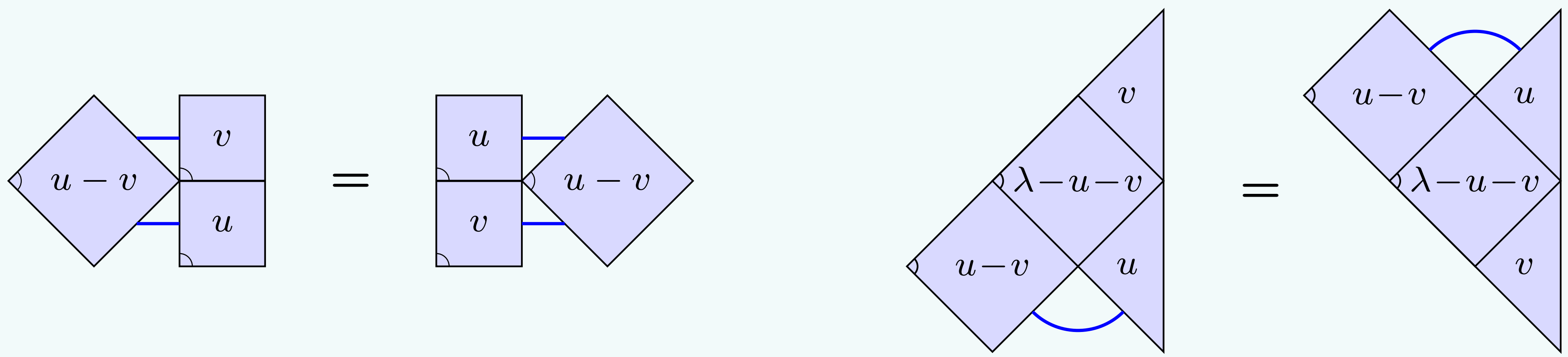
Logarithmic CFTs
(Logarithmic Minimal Models)



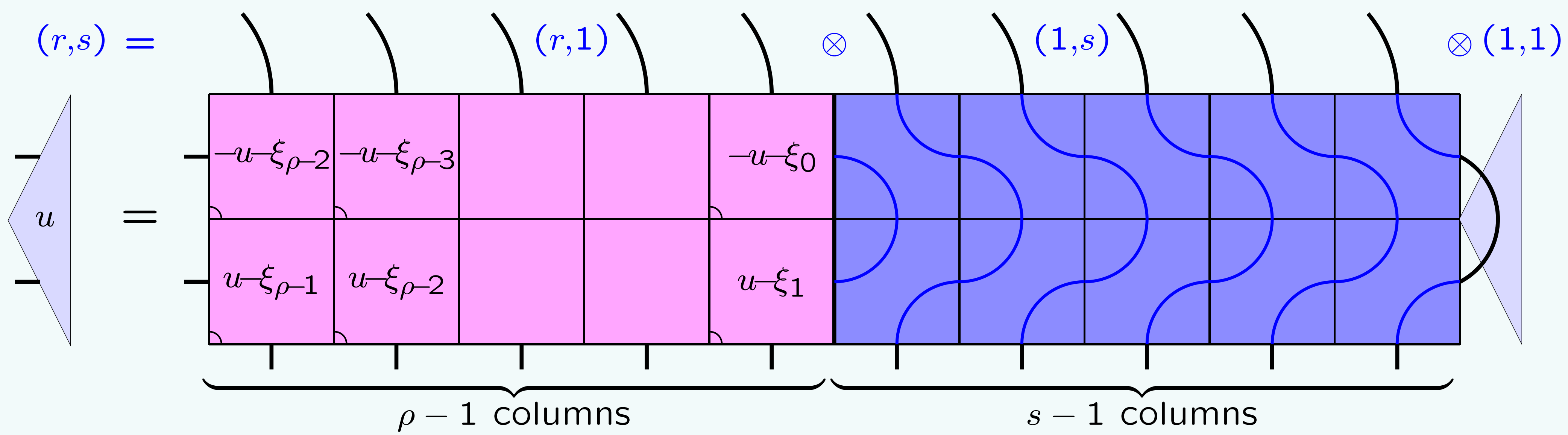
non-local degrees of freedom

Yang-Baxter Equations and Boundary Conditions

Yang-Baxter equations



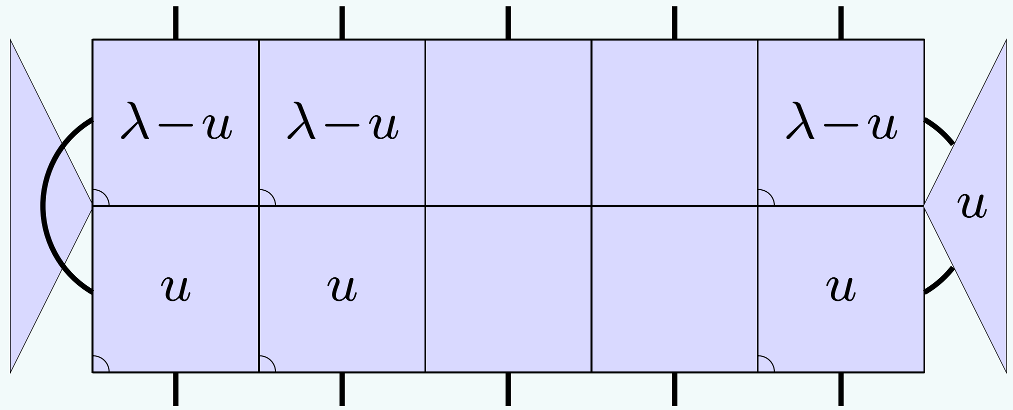
- (r,s)-**solution** $(r, s \in \mathbb{N}$, ρ is related to r, and ξ_k is linear in $\lambda)$



- Left boundary conditions are constructed similarly.

Double-Row Transfer Matrix and Link States

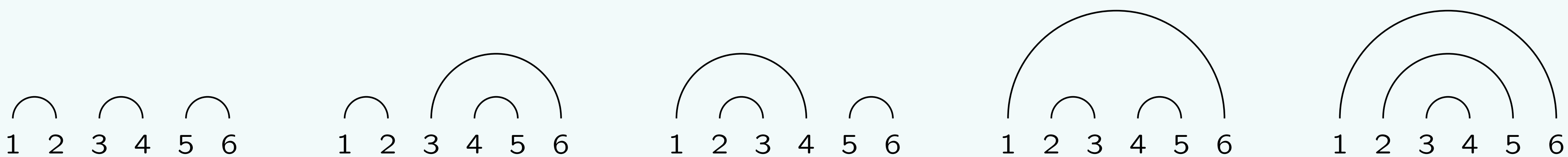
- For a strip with N columns, the double-row transfer "matrix" is the N-tangle

$\boldsymbol{D}(u)\ =$ 

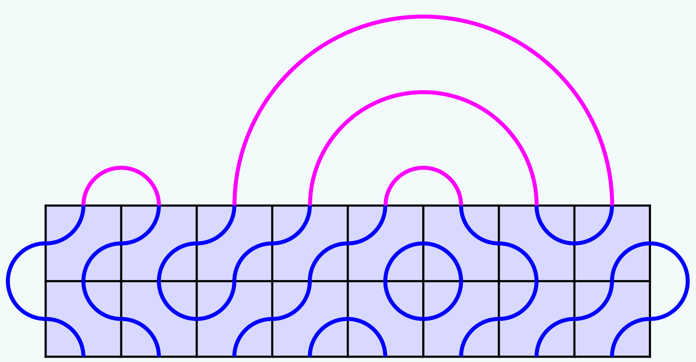
- **Matrix realizations** and their spectra are obtained by acting on **vector spaces**.

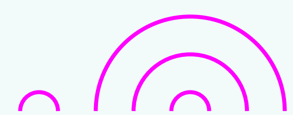
Link states

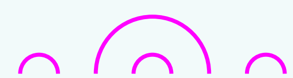
- An N-tangle acts on a vector space of **planar link diagrams**. Basis for $N = 6$:



Transfer matrix



instate: 

outstate: β^2 

Conformal Field Theory and Kac Representations

- With only **one** non-trivial (r,s)-type boundary condition, the double-row transfer matrix is found to be **diagonalizable**.

Continuum scaling limit

$$\boldsymbol{D}(u) \sim e^{-2u\mathcal{H}}, \qquad -\mathcal{H} \to L_0 - \frac{c}{24}, \qquad Z_{r,s}(q) = \mathrm{Tr}\, \boldsymbol{D}(u)^{M/2} \to q^{-c/24}\, \mathrm{Tr}\, q^{L_0} = \chi_{r,s}(q)$$

where q is the modular nome, $r,s \in \mathbb{N}$, while

$$c \;=\; 1 - \frac{6(p-p')^2}{pp'}, \qquad \chi_{r,s}(q) \;=\; q^{-c/24}\, \frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}, \qquad \Delta_{r,s} \;=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'}$$

- Associated to the boundary condition (r,s) is the so-called **Kac representation** (r,s).

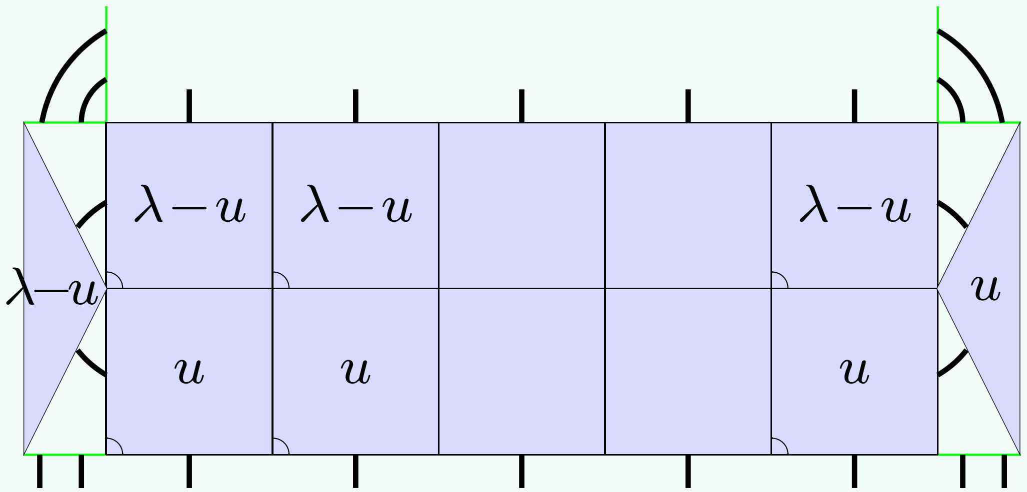
- As a representation of the Virasoro algebra, a Kac representation can be either (i) irreducible, (ii) reducible yet indecomposable, or (iii) fully reducible.

- There are **infinitely** many distinct Kac representations, associated with an infinitely extended Kac table.

- The **identity** representation is $(1,1)$. It is $\begin{cases} \text{irreducible,} & p=1 \\ \text{reducible yet indecomposable,} & p \geq 2 \end{cases}$

Lattice Implementation of Fusion

- **Fusion** is implemented on the lattice by taking non-trivial boundary conditions on the left and right $(r', s') \otimes (r, s)$:

$\boldsymbol{D}(u) =$ 

- In general, these fusion transfer matrices are **non-diagonalizable** as they can exhibit **non-trivial Jordan blocks**.

- In the continuum scaling limit, this non-diagonalizability gives rise to reducible representations $\mathcal{R}$ of **rank greater than 1** $\Rightarrow$ **Logarithmic CFT**.

- There are infinitely many of these higher-rank representations; all of rank 2 or 3.

- The fusion rules obtained empirically from the lattice are **commutative** and **associative**. For $\mathcal{LM}(1,2)$, they agree with Gaberdiel & Kausch (1996).

Classification of Kac Representations in $\mathcal{LM}(1,p')$

- From the lattice

$$(r,s) = (r,1) \otimes (1,s)$$

Conjecture: Combining the lattice approach with the Nahm-Gaberdiel-Kausch algorithm determines the module structure of (r,s) **upto** [$M_{\rho,\sigma}$ is irreducible]

$$(1,p'+1): \qquad M_{1,p'+1} \to M_{1,3p'-1} \qquad \textbf{vs} \qquad M_{1,p'+1} \leftarrow M_{1,3p'-1}$$

Assumption: The Kac representation $(1,p'+1)$ is a highest-weight representation (opt. I).

Conjectured classification: Kac representations are finitely generated Feigin-Fuchs modules

$$(r,s) = \begin{cases} Q^{\to}_{r,s}: \quad M_{k-r+1,p-s_0} \to M_{k-r+2,s_0} \leftarrow M_{k-r+3,p-s_0} \to \ldots \leftarrow M_{k+r-1,p-s_0} \to M_{k+r,s_0}, & 2r-1 < 2k \\ Q^{\leftarrow}_{r,s}: \quad M_{r-k,s_0} \leftarrow M_{r-k+1,p-s_0} \to M_{r-k+2,s_0} \leftarrow \ldots \leftarrow M_{r+k-1,p-s_0} \to M_{r+k,s_0}, & 2r-1 > 2k \end{cases}$$

where $s = s_0 + kp$ with $s_0 = 0,1,\ldots,p'-1$ and $k \in \mathbb{N}_0$, but $s \neq 0$.

Contragredient Kac representations $(r,s)^*$

$$\big\langle (r,s);\ r,s \in \mathbb{N} \big\rangle \simeq \big\langle (r,s)^*;\ r,s \in \mathbb{N} \big\rangle, \qquad \big\langle (r,s),(r,s)^*;\ r,s \in \mathbb{N} \big\rangle$$

Characters

$$\chi_{r,s}(q) = \chi^*_{r,s}(q) = \chi[Q_{r,s}](q), \qquad Q_{r,s} = V_{r,s}/V_{k+r+1,p-s_0}$$

$\mathcal{W}$-Extended Picture

$\mathcal{W}$-extended vacuum

$$(1,1)_{\mathcal{W}} := \lim_{n\to\infty} (2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = \bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,1)$$

Stability properties

$$\begin{aligned}(2m-1,s)\otimes(1,1)_{\mathcal{W}} &= (2m-1)\Big(\bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,s)\Big)\\ (2m,s)\otimes(1,1)_{\mathcal{W}} &= 2m\Big(\bigoplus_{n=1}^{\infty}2n\,(2n,s)\Big)\end{aligned}$$

$\mathcal{W}$-extended Kac representations

$$(r,s)_{\mathcal{W}} := \tfrac{1}{r}(r,s)\otimes(1,1)_{\mathcal{W}}, \qquad (r,s)_{\mathcal{W}} = \begin{cases}(1,s)_{\mathcal{W}}, & r \text{ odd}\\ \tfrac{1}{2}(2,s)_{\mathcal{W}}, & r \text{ even}\end{cases}$$

Elevated structure conjecture $\quad [n\cdot m = \tfrac{1}{2}(3-(-1)^{n+m})]$

$$(r,s_0+kp)_{\mathcal{W}}: \quad \underbrace{\widehat{M}_{2\cdot r\cdot k,s_0} \leftarrow \widehat{M}_{r\cdot k,p-s_0} \rightarrow \widehat{M}_{2\cdot r\cdot k,s_0} \leftarrow \ldots \leftarrow \widehat{M}_{r\cdot k,p-s_0} \rightarrow \widehat{M}_{2\cdot r\cdot k,s_0}}_{\#=2k+1}$$

$\mathcal{WLM}(1,2)$ examples $\quad$ [The $\mathcal{W}$-irreducible $\widehat{M}_{r,s}$ is represented by its conformal weight $\Delta_{r,s}$]

$$(1,3)_{\mathcal{W}}: \begin{array}{ccccc}1 & & & & 1\\ & \nwarrow & & \nearrow & \\ & & 0 & & \end{array} \qquad (2,3)_{\mathcal{W}}: \begin{array}{ccccc} & & 1 & & \\ & \swarrow & & \searrow & \\ 0 & & & & 0\end{array} \qquad (1,5)_{\mathcal{W}}: \begin{array}{ccccccccc} & & 1 & & & & 1 & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ 0 & & & & 0 & & & & 0\end{array}$$

Polynomial Fusion Ring

Proposition: The contragrediently extended $\mathcal{W}$-Kac fusion algebra is isomorphic to the polynomial ring generated by X, Y, Z and Z^* modulo the ideal

$$\mathcal{I} = \Big(X^2 - 1, P_p(X,Y), Q_p(Y,Z), Q_p(Y,Z^*), R_p(Y,Z,Z^*)\Big)$$

that is,

$$\boxed{\Big\langle (r,s)_{\mathcal{W}}, (r,s)^*_{\mathcal{W}};\ r,s\in\mathbb{N}\Big\rangle \simeq \mathbb{C}[X,Y,Z,Z^*]/\mathcal{I}}$$

where

$$\begin{array}{rcl} P_p(X,Y) &=& \Big[X - T_p(\tfrac{Y}{2})\Big]U_{p-1}(\tfrac{Y}{2}) \\ Q_p(Y,Z) &=& \Big[Z - U_p(\tfrac{Y}{2})\Big]U_{p-1}(\tfrac{Y}{2}) \end{array} \qquad R_p(Y,Z,Z^*) = ZZ^* - U_p^2(\tfrac{Y}{2})$$

Here $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively.

For $r\in\mathbb{Z}_{1,2}$, $b\in\mathbb{Z}_{0,p-1}$ and $k\in\mathbb{N}_0$, the isomorphism reads

$$\begin{array}{rcl} (r,b+kp)_{\mathcal{W}} &\leftrightarrow& X^{r-1}\Big(U_{kp+b-1}(\tfrac{Y}{2}) + \Big[Z^k - U_p^k(\tfrac{Y}{2})\Big]U_{b-1}(\tfrac{Y}{2})\Big) \\ (r,b+kp)^*_{\mathcal{W}} &\leftrightarrow& X^{r-1}\Big(U_{kp+b-1}(\tfrac{Y}{2}) + \Big[(Z^*)^k - U_p^k(\tfrac{Y}{2})\Big]U_{b-1}(\tfrac{Y}{2})\Big) \\ \hat{\mathcal{R}}_r^b &\leftrightarrow& (2-\delta_{b,0})X^{r-1}T_b(\tfrac{Y}{2})U_{p-1}(\tfrac{Y}{2}) \end{array}$$

Summary, Further Results and Outlook

- Infinite series of Yang-Baxter integrable lattice models of non-local statistical mechanics.
- Description in terms of planar Temperley-Lieb algebras.
- Logarithmic CFTs with infinitely many (higher-rank) indecomposable representations.
- $\mathcal{W}$-extended picture with (in)finitely many indecomposable representations.
- Fusion rules and polynomial fusion rings for $\mathcal{LM}(p,p')$ and $\mathcal{WLM}(p,p')$.

- Exact solution for critical dense polymers $\mathcal{LM}(1,2)$ on the strip.
- Verlinde-like formulas from spectral decompositions.
- Links to stochastic Loewner evolution.
- Exact solution for critical dense polymers on the cylinder.
- Coset graphs and modular invariant partition functions. [Talk by Paul Pearce]

- Exact solutions for other models, in particular critical percolation described by $\mathcal{LM}(2,3)$.
- Open boundary conditions.
- Dilute polymers and generalizations thereof.
- Lattice interpretation of $\mathcal{W}$-symmetry.
- From planar TL algebras to more general diagram algebras such as the BMW algebras.