Wrapping corrections beyond the $\mathfrak{sl}(2)$ sector in $\mathcal{N} = 4$ SYM

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Based on

- M. Beccaria, G. Macorini, C. Ratti

  Wrapping corrections, reciprocity and BFKL beyond the $\mathfrak{sl}(2)$ subsector in $\mathcal{N} = 4$ SYM

  JHEP 1106 (2011) 071 [arxiv:1105.3577 [hep-th]]

- M. Beccaria, G. Macorini, C. Ratti

  Large spin expansion of the wrapping correction to Freyhult-Rej-Zieme twist operators

  JHEP 1108 (2011) 050 [arXiv:1107.2568 [hep-th]]

Introduction and motivations

Computation of anomalous dimensions of operators in $\mathcal{N} = 4$ SYM

$$\Updownarrow$$

Diagonalization of H of associated spin chains

$$\Updownarrow$$

Bethe Ansatz from the analysis of the scattering of magnons along the chain

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Required: asymptotic region on the chain

Problematic when the interaction range of H is longer than the same chain!

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Finite size corrections $\longrightarrow$ wrapping corrections

$$\gamma = \underbrace{\gamma_A}_{\text{asymptotic}} + \underbrace{\gamma_W}_{\text{wrapping}}$$

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Integrability of the planar theory carries to the definition of

- TBA equations for the $AdS_5 \times S^5$ superstring for the $Y_{a,s}(u)$–functions

[Arutyunov et al. 07]

$$\gamma = \sum_i \epsilon_1(u_{4,i}) + \sum_{a \geq 1} \int_{\mathbb{R}} \frac{du}{2\pi i} \frac{\partial \epsilon_a^\star}{\partial u} \log(1 + Y_{a,0}^\star(u))$$

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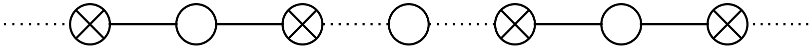
- From the TBA equations $\longrightarrow$ Y–system to computate the Y's

[Gromov, Kazakov, Vieira 09]

$\Longrightarrow$ Complete spectrum of operators in planar $\mathcal{N}=4$ SYM available

- at any value of the coupling λ
- for any operator in of the $\mathfrak{psu}(2,2|4)$ algebra of $\mathcal{N}=4$ SYM



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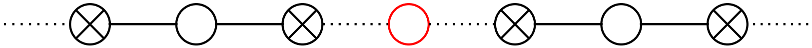

$\Longrightarrow$ Tests of the Y–system mainly in the $\mathfrak{sl}(2)$ closed sector

$$\mathbb{O}^{\mathcal{Z}}_{N,L} = \mathrm{Tr}\left(\mathcal{D}^{s_1}\mathcal{Z}\cdots\mathcal{D}^{s_L}\mathcal{Z}\right)+\cdots \qquad N = s_1+\cdots+s_L$$

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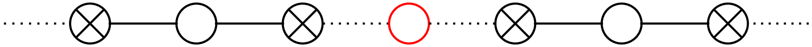

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Most advanced results: closed formulas in the spin N valid at

- 5 loop for twist $L=2$ operators [Lukowski, Rej, Velizhanin 09]
- 6 loop for twist $L=3$ operators [Velizhanin 10]

Properties of the results in the $\mathfrak{sl}(2)$ sector:

- Generalized Gribov–Lipatov reciprocity at large N respected

$$\gamma = \sum_s \frac{a_s \, logJ}{J^{2s}}, \qquad J^2 = N(N+c)$$

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- For twist $L = 2$ operators: BFKL equations satisfied by the poles for $N < 0$

$$\begin{aligned} \frac{\omega}{-g^2} &= \chi_0\left(\frac{\gamma}{2}\right) \\ \chi_0(x) &= S_1(x) + S_1(-x-1) \\ \omega &= N + 1 \ll 1 \end{aligned}$$

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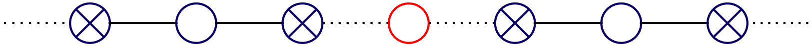
What happens for operators in the more general $\mathfrak{psu}(2,2|4) \supset \mathfrak{sl}(2)$?



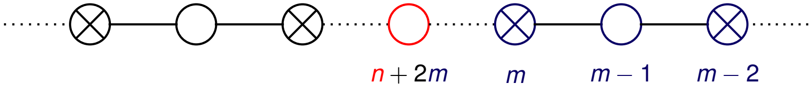
Freyhult–Rej–Zieme operators

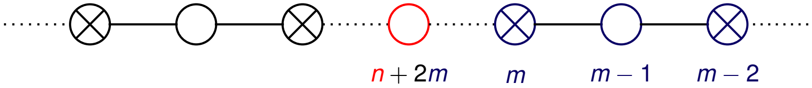
$$\mathbb{O}^{\mathrm{FRZ}}_{n,m} = \mathrm{Tr}\,(\mathscr{D}^{n+m}\bar{\mathscr{D}}^{m}\mathcal{Z}^{3}) + \cdots$$

Two spins, twist-3 operators:

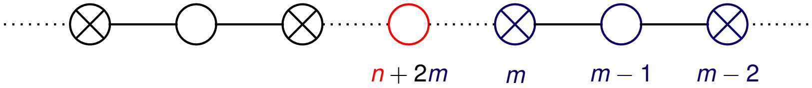
$$S_1 = n + m - \frac{1}{2} \qquad S_2 = m - \frac{1}{2}$$

Strong coupling dual: Spinning string with S_1 and S_2 spins in AdS_5

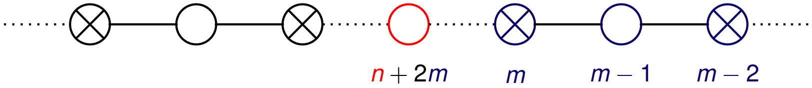

$n + 2m$
m
$m - 1$
$m - 2$



- $m = 0 \longrightarrow$ twist-3 operators in the $\mathfrak{sl}(2)$ sector



- $m = 0 \longrightarrow$ twist-3 operators in the $\mathfrak{sl}(2)$ sector
- $m = 1 \longrightarrow$ descendant of twist-2 operators



- $m=0 \longrightarrow$ twist-3 operators in the $\mathfrak{sl}(2)$ sector
- $m=1 \longrightarrow$ descendant of twist-2 operators
- $m=2 \longrightarrow$ 3–gluon operators

They can be written as

$$\mathbb{O}^{A}_{n,2} = \mathrm{Tr}\left[\partial_+^{n_1}\mathcal{A}(0)\,\partial_+^{n_2}\mathcal{A}(0)\,\partial_+^{n_3}\mathcal{A}(0)\right], \qquad n = n_1 + n_2 + n_3$$

Anomalous dimensions of 3–gluons operators $\mathbb{O}^{A}_{n,m=2}$

$$\gamma = \sum_{k=1}^{\infty} g^{2k}\gamma_{A,k} + g^{2k+6}\gamma_{W,k}$$

- $\gamma_{A,k}$ known up to $k = 4$

[Beccaria 07, Beccaria, Forini 08]

- In general

$$\gamma_{A,k\le 4} = \sum_{h+\ell=k} \frac{S_{k,\ell}(n)}{(n+1)^h}$$

Maximum trascendentality principle violated

Properties of $\gamma_{A,k\leq 4}$

- *Reciprocity respecting* structure at large n

$$\gamma_{A,k\leq 4} = \sum_s \frac{a_s \, logJ}{J^{2s}}, \qquad J^2 = n(n+8)$$

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- *BFKL–like behaviour* for the poles around negative values of n up to $k = 3$

$$\frac{\omega}{-g^2} = 2\,\chi_1\left(\frac{\gamma}{2}\right)$$

$$\begin{aligned} \chi_1(x) &= \chi_0(x) + \chi_0(x+1) \\ \chi_0(x) &= S_1(x) + S_1(-x-1) \\ \omega &= n+1 \ll 1 \end{aligned}$$

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  [Freyhult, Rej, Zieme 09]
- *Reciprocity respecting* structure still present in $\gamma_{A,k}$

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- $\gamma_{A,k}$ known in the limit of large n and $\frac{n}{m}$ or m fixed [Freyhult, Rej, Zieme 09]
- *Reciprocity respecting* structure still present in $\gamma_{A,k}$

Mandatory to study at least $\gamma_{W,1}$!

- $m = 2 \longrightarrow$ closed formulas in n found also for the wrapping [BMR 1105.3577]
- $m > 2 \longrightarrow$ results in the limit of large n, fixed m [BMR 1107.2568]

Y–system for leading wrapping

Y–system:

- System of functional equations for the functions $Y_{a,s}(u)$

$$\frac{Y_{a,s}^{+}\,Y_{a,s}^{-}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

- For the wrapping corrections

$$\gamma_W = \sum_{a\geq 1}\int_{\mathbb{R}} \frac{du}{2\pi i}\,\frac{\partial \epsilon_a^\star}{\partial u}\,\log(1+Y_{a,0}^\star(u))$$

- Related to the transfer matrices and the Hirota equation through

$$Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$

- Solution

$$Y_{a,0}(u) \simeq \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^{L} \frac{\phi^{[-a]}}{\phi^{[+a]}} T^{\mathrm{L}}_{a,1} \, T^{\mathrm{R}}_{a,1}$$

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Simplifications for the first order corrections:

- $$\gamma_{W,1} = -\frac{1}{\pi} \sum_{a \geq 1} \int_{\mathbb{R}} du\; Y^{\star}_{a,0}(u)$$

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Simplifications for the first order corrections:

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$$\gamma_{W,1} = -\frac{1}{\pi}\sum_{a\geq 1}\int_{\mathbb{R}} du\; Y^{\star}_{a,0}(u)$$

- Closed formulas in a for dispersion, fusion factor and transfer matrices

$$\left(\left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L\right)^* = \left(\frac{4g^2}{a^2+4u^2}\right)^L$$

$$\left(\frac{\phi^{[-a]}}{\phi^{[+a]}}\right)^* = [(Q_4^+(0)]^2\, \frac{Q_4^{[1-a]}}{Q_4^{[-1-a]}\,Q_4^{[a-1]}\,Q_4^{[a+1]}}\, \frac{Q_5^{[-a]}}{Q_5}$$

$$T^{*,L}_{a,1} = (\log Q_4)' \Big|_{u=-i/2}^{u=+i/2} g^2 \frac{(-1)^a}{Q_4^{[1-a]}} \sum_{\substack{k=-a \\ \Delta k=2}}^{a} \frac{Q_4^{[-1-k]} - Q_4^{[1-k]}}{u - i\frac{k}{2}} \Big|_{Q_4^{[-1-a]}, Q_4^{[-1-a]} \to 0}$$

$$T^{*,\mathrm{R}}_{a,1} = (-1)^{a+1} \frac{Q_5^{[a]} Q_7^{[-a]}}{Q_4^{[1-a]}} \sum_{\substack{k=1-a \\ \Delta k=2}}^{a-1} \frac{Q_4^{[k]}}{Q_6^{[k]}} \left(\frac{Q_6^{[k+2]} - Q_6^{[k]}}{Q_5^{[k+1]} Q_7^{[k+1]}} + \frac{Q_6^{[k-2]} - Q_6^{[k]}}{Q_5^{[k-1]} Q_7^{[k-1]}} \right)$$

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Crucial ingredients: 1–loop Baxter polynomials Q_i

$\longrightarrow$ Known exactly for both

- ♠ $m = 2$ [Beccaria 07]
- ♠ $m > 2$ [Freyhult, Rej, Zieme 09]

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$\longrightarrow$ Complicated hypergeometric functions in n
Reduced to polynomials in u once n is fixed

3–gluon operators

Compute the first $\gamma_{W,1,n}$

$$\begin{aligned}
\gamma_{W,1,n=2} &= \frac{5348840}{2187} + \frac{528640}{81}\,\zeta_3 - \frac{89600}{9}\,\zeta_5 \\
\gamma_{W,1,n=4} &= \frac{1216307603}{331776} + \frac{235081}{24}\,\zeta_3 - 14910\,\zeta_5 \\
\gamma_{W,1,n=6} &= \frac{4018092206843}{810000000} + \frac{24863234}{1875}\,\zeta_3 - \frac{100848}{5}\,\zeta_5 \\
\gamma_{W,1,n=8} &= \frac{27624795456401}{4374000000} + \frac{56992078}{3375}\,\zeta_3 - \frac{231088}{9}\,\zeta_5 \\
\gamma_{W,1,n=10} &= \frac{206037054943950841}{26682793200000} + \frac{3718258258}{180075}\,\zeta_3 - \frac{1538160}{49}\,\zeta_5 \\
\gamma_{W,1,n=12} &= \frac{143200464784761259303}{15613245849600000} + \frac{80764059527}{3292800}\,\zeta_3 - \frac{521985}{14}\,\zeta_5 \\
\gamma_{W,1,n=14} &= \frac{7678365619317330991462 93}{720270745447680000 00} + \frac{32079436339187}{1125211500}\,\zeta_3 - \frac{24571256}{567}\,\zeta_5 \\
\gamma_{W,1,n=16} &= \frac{185723339127402837048 7}{152438253004800000} + \frac{77595407257}{2381400}\,\zeta_3 - \frac{148580}{3}\,\zeta_5 \\
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&\dots
\end{aligned}$$

and find a closed formula in n to fit these numbers

$$\gamma_{W,1,n} = \left(r_{0,n} + r_{3,n}\,\zeta_3 + r_{5,n}\,\zeta_5\right)\,g^8$$

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Being inspired by

- The 1-loop asymptotic anomalous dimension $\gamma_{A,1} = \left(4\,S_1 + \frac{2}{n+1} + 4\right)$
- The general structure of the known asymptotic anomalous dimensions

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we propose

$$r_{5,N} = 80\left(4\,S_1 + \frac{2}{n+1} + 4\right)\left(-4(n+1) + \frac{1}{n+1}\right)$$

$$r_{3,N} = 16\left(4\,S_1 + \frac{2}{n+1} + 4\right)\times$$
$$\left[8\,(n+1)\,S_2 + 8 + \frac{2}{n+1}(2 - S_2) - \frac{2}{(n+1)^2} - \frac{1}{(n+1)^3}\right]$$

$$r_{0,N} = 2\left(4\,S_1 + \frac{2}{n+1} + 4\right)\times$$
$$\left[16\,(n+1)\,(2\,S_{2,3} - S_5) + 32\,S_3 + \frac{4}{n+1}(S_5 - 2S_{2,3} + 4S_3) + \right.$$
$$\left. + \frac{8}{(n+1)^2}(-S_3 + 2) + \frac{4}{(n+1)^3}(-S_3 + 4) - \frac{4}{(n+1)^5} - \frac{1}{(n+1)^6}\right]$$

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$$+\frac{8}{(n+1)^2}(-S_3 + 2) + \frac{4}{(n+1)^3}(-S_3 + 4) - \frac{4}{(n+1)^5} - \frac{1}{(n+1)^6}\Big]$$

These formulas directly checked up to $n = 70$!

Two nice properties of this Ansatz

1) Reciprocity
For large n and with $J^2 = n(n+8)$, we get

$$
\begin{aligned}
\widetilde{r}_{5,N} &= -320J - \frac{80}{J} + \frac{10}{J^5} - \frac{25}{2\,J^7} + \mathcal{O}\left(\frac{1}{J^8}\right) \\
\widetilde{r}_{3,N} &= 128\zeta_2 J + \frac{32\zeta_2}{J} - \frac{64}{3J^2} + \frac{464}{15J^4} - \frac{4\zeta_2}{J^5} - \frac{4688}{105J^6} + \frac{5\zeta_2}{J^7} + \mathcal{O}\left(\frac{1}{J^8}\right) \\
\widetilde{r}_{0,N} &= (320\zeta_5 - 128\zeta_2\zeta_3)\,J + \frac{80\zeta_5 - 32\zeta_2\zeta_3}{J} - \frac{32\,(\zeta_3 - 1)}{3J^2} + \frac{\frac{232\zeta_3}{15} - \frac{352}{15}}{J^4} + \\
&\quad + \frac{4\zeta_2\zeta_3 - 10\zeta_5}{J^5} + \frac{\frac{4834}{105} - \frac{2344\zeta_3}{105}}{J^6} + \frac{\frac{25\zeta_5}{2} - 5\zeta_2\zeta_3}{J^7} + \mathcal{O}\left(\frac{1}{J^8}\right)
\end{aligned}
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1) Reciprocity
For large n and with $J^2 = n(n+8)$, we get

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Summing

$$\zeta_5\,\widetilde{r}_{5,N} + \zeta_3\,\widetilde{r}_{3,N} + \widetilde{r}_{0,N} = \frac{\frac{32}{3} - 32\zeta_3}{J^2} + \frac{\frac{232\zeta_3}{5} - \frac{352}{15}}{J^4} + \frac{\frac{4834}{105} - \frac{2344\zeta_3}{35}}{J^6} + \mathcal{O}\left(\frac{1}{J^8}\right)$$

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2) BFKL–like behaviour up to $k = 4$:
Expanding at weak coupling the BFKL–like equation $\frac{\omega}{-g^2} = 2\,\chi_1\left(\frac{\gamma}{2}\right)$

$$\begin{aligned}\gamma \quad = \quad & \left(-\frac{4}{\omega} + \dots\right) g^2 + \left(\frac{8}{\omega^2} + \dots\right) g^4 + \left(\frac{\mathbf{0}}{\omega^3} + \dots\right) g^6 + \\ & + \left(-\frac{32\,(1+2\,\zeta_3)}{\omega^4} + \dots\right) g^8 + \left(\frac{512\,\zeta_3}{\omega^5} + \dots\right) g^{10} + \dots\end{aligned}$$

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\gamma_{W,1} &= \frac{4}{\omega^7}-\frac{24}{\omega^6}+\frac{32-\frac{4\pi^2}{3}}{\omega^5}+\frac{-24\zeta_3+\frac{16\pi^2}{3}+64}{\omega^4}+\dots
\end{aligned}$$

FRZ operators

For $m > 2$ just the large n expression for γ_A known

$\longrightarrow$ we look for $\gamma_{W,1}$ in the same limit

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$$\gamma_{W,1} = -\frac{1}{\pi}\sum_{a=1}^{\infty}\int_{\mathbb{R}} du\, Y^{\star}_{a,0} = -2i\sum_{a=1}^{\infty} \underset{u=i\frac{a}{2}}{\mathrm{Res}}\; Y^{\star}_{a,0}$$

A careful analysis allow to write $\lim_{n\to\infty} \mathrm{Res}_{u=i\frac{a}{2}}\; Y^{\star}_{a,0}$ as linear combinations of

- a–polynomially dependent coefficients
- Logarithmic derivatives of specific hypergeometric functions
  $\longrightarrow$ their large n limit computed with standard techniques

[Beccaria, Catino 07]

Example: $m = 3$
for $a \geq 4$

$$\operatorname*{Res}_{u=i\frac{a}{2}} Y^{\star}_{a,0} = \left(\frac{1}{n^2} - 11\,\frac{1}{n^3}\right)\left[L'(i) - \frac{11i}{2}\right] f_1(a) + \frac{1}{n^4}[L'(i)\, f_2(a) + f_3(a)] + \ldots$$

$$\begin{aligned}
f_1(a) &= -\frac{12(2a-1)\left(3a^2-3a-4\right)}{(a-2)^2(a-1)^3a^3(a+1)^2} \\
f_2(a) &= -\frac{12}{(a-3)^2(a-2)^3(a-1)^3a^3(a+1)^3} \times \left(20a^8 + 426a^7 - 4399a^6 + \right. \\
&\qquad \left. 12288a^5 - 6156a^4 - 19036a^3 + 17131a^2 + 8538a - 6108\right) \\
f_3(a) &= \frac{6i}{(a-3)^2(a-2)^3(a-1)^3a^3(a+1)^3} \times \left(220a^8 + 4734a^7 - 48797a^6 + \right. \\
&\qquad \left. 136256a^5 - 68196a^4 - 211220a^3 + 190033a^2 + 94734a - 67764\right)
\end{aligned}$$

$$\gamma_{W,1,m=2} = -\frac{512}{3}\,(3\,\zeta_3 - 1)\,\frac{3\,\log\frac{\bar{n}}{2} + 1}{n^2} + \frac{2048}{3}\,(3\,\zeta_3 - 1)\,\frac{2\,\log\frac{\bar{n}}{2} + 1}{n^3} + \\ -\frac{1536}{5}\,(77\,\zeta_3 - 24)\,\frac{\log\frac{\bar{n}}{2}}{n^4} + \dots$$

$$\gamma_{W,1,m=3} = -\frac{4}{3}\,(36\,\zeta_3 + 5\pi^2 - 3)\,\frac{2\,\log\bar{n} + 1}{n^2} + \frac{88}{3}\,(36\,\zeta_3 + 5\pi^2 - 3)\,\frac{\log\bar{n}}{n^3} + \\ -\frac{2}{9\,n^4}\left[4\,(9108\,\zeta_3 + 1289\pi^2 - 615)\,\log\bar{n} - 14868\,\zeta_3 - 2017\pi^2 + 1527\right] + \dots$$

$$\gamma_{W,1,m=4} = -\frac{1024}{1215}\,(81\,\zeta_3 - 32)\,\frac{3\,\log\frac{\bar{n}}{2} + 4}{n^2} + \frac{7168}{1215}\,(81\,\zeta_3 - 32)\,\frac{6\,\log\frac{\bar{n}}{2} + 5}{n^3} + \\ -\frac{1024}{8505\,n^4}\left[138240\,(1971\,\zeta_3 - 760)\,\log\frac{\bar{n}}{2} + 143289\,\zeta_3 - 53248\right] + \dots$$

Comparison of the results with a numerical estimate*

$m = 3$

n	estimate	(LO	NLO	NNLO)	full expansion	diff %
10	-1.85741	$(-4.03873$	3.78537	$-2.54806)$	-2.80142	51%
30	-0.43878	$(-0.59461$	0.19368	$-0.04535)$	-0.44628	1.7%
50	-0.19727	$(-0.23847$	0.04720	$-0.00671)$	-0.19798	0.36%

$m = 4$

n	estimate	(LO	NLO	NNLO)	full expansion	diff %
10	-1.20146	$(-2.90878$	3.49384	$-3.57612)$	-2.99106	149%
30	-0.29376	$(-0.42407$	0.17647	$-0.06188)$	-0.30948	5.4%
50	-0.13424	$(-0.16955$	0.04284	$-0.00909)$	-0.13579	1.2%

* we plot $-Im\left(\frac{\gamma}{2}\right)$

Substituting in the results

$$J^2 = n(n + 3m + 2)$$

$$\begin{aligned}
\gamma_{W,1,m=2} &= -\frac{256}{3J_2^2}\,(3\,\zeta_3 - 1)\left(\log\frac{\bar{J_2}^2}{4} + 2\right) + \\
&+ \frac{256}{15J_2^4}\left[(267\,\zeta_3 - 104)\,\log\frac{\bar{J_2}^2}{4} + 480\,\zeta_3 - 160\right], \\
\gamma_{W,1,m=3} &= -\frac{4}{3J_3^2}\,(36\,\zeta_3 + 5\pi^2 - 3)\left(\log\bar{J_3}^2 + 1\right) + \\
&+ \frac{4}{9J_3^4}\left[2\,(1980\,\zeta_3 + 263\pi^2 - 237)\,\log\bar{J_3}^2 + 900\,\zeta_3 + 101\pi^2 - 219\right], \\
\gamma_{W,1,m=4} &= -\frac{512}{1215J_4^2}\,(81\,\zeta_3 - 32)\left(3\,\log\frac{\bar{J_4}^2}{4} + 8\right) + \\
&+ \frac{512}{8505J_4^4}\left[(67311\,\zeta_3 - 29112)\,\log\frac{\bar{J}_4^2}{4} + 102384\,\zeta_3 - 47168\right].
\end{aligned}$$

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Conclusions

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  $\longrightarrow$ beyond the $\mathfrak{sl}(2)$ subsector of $\mathcal{N} = 4$ SYM theory

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  - General formula for the wrapping valid for any value of the spin n
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- For more general FRZ operators ($m > 2$)
  - Large n behaviour for m fixed
  - Reciprocity checked