Boundary matrix model and Liouville Gravity

Chaiho Rim (Sogang University)

8th Bologna Workshop on CFT and Integrable models

In collaboration with A. Belavin, J-E. Bourgine, G. Ishiki

1001.4356 (PLB 687), 1006.3906 (PLB 694)
1010.1363 (JHEP 1012), 1012.1467 (PLB 698)
1107.4186 and more to appear

Inspired by Al. Zamoldchikov hep-th/0505063, 0508044

1. Matrix model describes the discretized 2d space-time, fluctuating space-time interacting with matter. At the continuum limit, the matrix model describes quantum gravity coupled with matter

2. Continuum theory is described by Liouvill theory coupled to matter.

3. Bulk interaction is well known from both of the approaches.

4. Our goal is to describe the boundary effect from the matrix model approach

Plan of Talk

1. Brief introduction: Matrix model and Liouville gravity

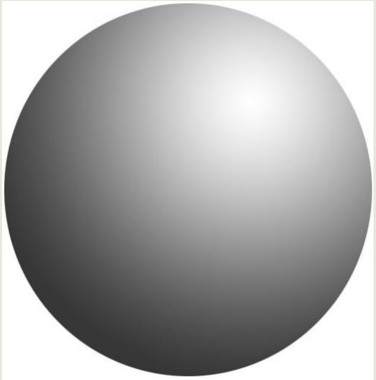
2. Boundary description of matrix model

3. Boundary and bulk correlations

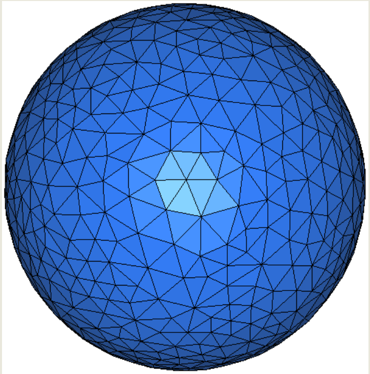
4. Summary and remarks

Matrix model: discrete approach to space-time

Key Idea : Discretization of 2D surface (world sheet)



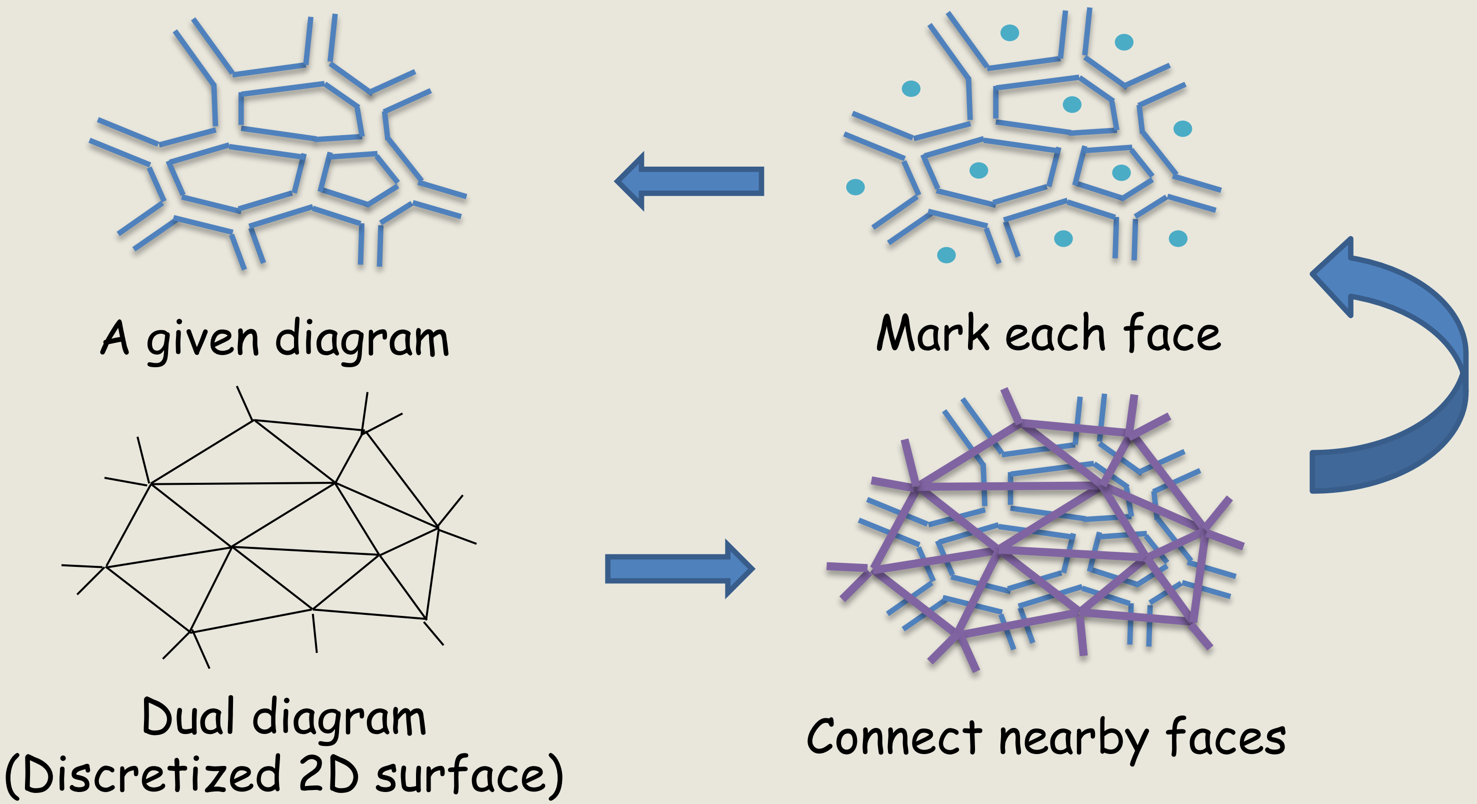
Smooth 2d surface



Lattice theory

- Lattice theory for gravity = Fluctuating lattice ('Dynamical lattice')
- Matrix model = dual picture of the dynamical lattice

- A Feynman diagram as a discretized surface.



Matrix model

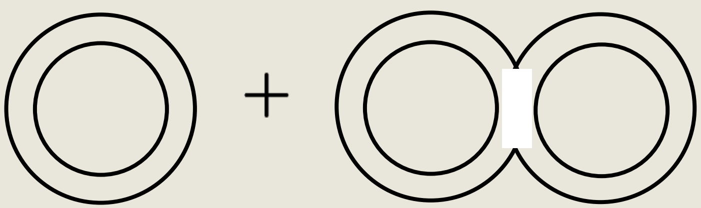
- One matrix model $\Leftrightarrow$ (2,2p+1) Minimal Liouville gravity $\quad b^2 = \dfrac{2}{2p+1}$

$$e^{Z_{MM}} = \int DM e^{-\frac{N}{g}\mathrm{tr}V(M)}$$

$$V(M) = \tfrac{1}{2}M^2 + \tfrac{v_4}{4!}M^4 + \cdots \qquad M: \quad N \times N \text{ Hermitian matrix}$$

- 1/N expansion = Sum of topology .

$$z_{MM} = \sum_{\substack{\text{connected}\\ \text{Feynman}\\ \text{diagram}}} =$$



$+ \cdots$

- Sum of topology. $$Z_{\mathrm{MM}} = \sum_{g=0}^{\infty} a_g N^{\chi_g}$$

$\chi_g = 2 - 2g$: Euler characteristic

- Double scaling limit is needed to describe the fluctuating surface

$$\left(\begin{array}{l} N \to \infty, \quad g \to g_c \\ \text{with} \quad N(g-g_c)^{1-\gamma/2} \quad \text{fixed} \end{array} \right)$$

$$\Longrightarrow \quad Z_{MM} \to Z_{\text{2d gravity+matter}}$$

Liouville gravity as a continuum theory of 2d gravity

2d gravity is described in terms of genus expansion

Liouville field , ghost and matter (Polyakov).

$$Z = \int DgDXe^{-S[g,X]}$$

$$= \sum_{\text{topology}} \int D\phi DXe^{-S_L[\phi]+\text{matter CFT}+\text{ghost}}$$

ϕ : Liouville field

(Minimal) Liouville gravity

- gravity as Liouville action

$$S_L[\phi] = \frac{1}{4\pi}\int_{\mathcal{M}} dx^2 \sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a\phi\partial_b\phi + QR[\hat{g}]\phi + 4\pi\mu e^{2b\phi}\right)$$

$Q = b + \frac{1}{b}$ $\quad c_\phi = 1 + 6Q^2$ $\quad \mu$:Bulk cosmological constant

- matter part : use minimal CFT for simplicity

The total central charge vanishes: $c_\phi + c_{\text{matter}} + c_{\text{ghost}} = 0$

$q = b - \frac{1}{b}$ $\quad c_{\text{matter}} = 1 - 6q^2$ $\quad c_{\text{ghost}} = -26$

Liouville gravity

- add interaction : maintaining the conformal symmetry

$$Z = \int DgDXe^{-S[g,X]}$$

$$= \sum_{\text{topology}} \int D\phi DXe^{-S_L[\phi]+\text{matter CFT}+\text{ghost}}$$

$$+ \sum_{mn} \lambda_{mn} \int_M \Phi_{mn} e^{\alpha_{mn}\phi}$$

ϕ : Liouville field, Φ_{mn} : Matter field

p-critical model : $b = \sqrt{\frac{2}{2p+1}}$

Relation between two different frames (Resonance relation)

- $$e^{Z_{\mathrm{OMM}}} = \int DM e^{-\frac{N}{g}\mathrm{tr}V(M)+\sum_{n=1}^{p} t_n \sigma_n}$$

  t_n : coupling to scaling op. KdV frame.

- $$Z_{(2,2p+1)\ \mathrm{MG}} = \int DgDX e^{-S[g,X]+\sum_{n=1}^{p} \lambda_n O_{1n}}$$

  λ_n : coupling to (1,n) operator with gravity dressing. CFT frame.

- In general $$\langle O_{1i} O_{1j} \rangle = \left. \frac{\partial^2 Z_{(2,2p+1)\mathrm{MG}}}{\partial \lambda_i \partial \lambda_j} \right|_{\lambda_k=0} \neq \left. \frac{\partial^2 Z_{\mathrm{OMM}}}{\partial t_i \partial t_j} \right|_{t_k=0}$$

Moore, Seiberg, Staudacher (1991)

What is $t_k = t_k(\{\lambda_j\})$?

Lee-Yang case (p=2): Al. Zamolodchikov (2005)

- For (2, 2p+1) minimal gravity on sphere, Belavin and Zamoldchikov (2008)

  - 1-point function is 0 (except (1,1) operator)
  - 2-point function is orthogonal.
  - 3-point function is 0 unless…

$$P(u,\mu,\{\lambda_k\}) \equiv P(u,\mu,\{t_k(\lambda)\}) = u_0^{p+1} Q(u/u_0,\{\lambda_k\})$$

$$u_0 \propto \sqrt{\mu}$$

$$Q(x,\{\lambda_k\}) = \sum_{n=0}^{\infty} \sum_{k_1 \ldots k_n = 1}^{p-1} \frac{\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n}}{n!} \left(\frac{d}{dx}\right)^{n-1} L_{p-\sum k_i - n}(x),$$

$$Z(\mu,\{\lambda_k\}) = \frac{1}{2}\int_0^{u_*} Q^2(u/u_0,\lambda_k)\,du$$

Boundary description of matrix model

Boundary Liouville gravity

- In the presence of boundary, boundary Liouville action is added

$$+\frac{1}{2\pi}\int_{\partial\mathcal{M}} d\xi \hat{g}^{1/4}\left(QK[\hat{g}]\phi + 2\pi\mu_b e^{b\phi}\right)$$

μ_b : Boundary cosmological constant

- Matter field also lives at the boundary and couples to the gravity

Boundary object

$$B_{m,n}^{(s_1;l_1)(s_2,l_2)} = \int_{\partial M} \left[\Phi_{mn} e^{\beta_{mn}\phi}\right]^{(s_1,l_1)(s_2,l_2)}$$

(m, n) primary field

Liouville dressing

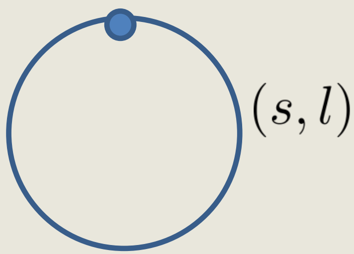
boundary conditions

Gravity side: $\mu_b(s) \sim \sqrt{\mu}\cosh(\pi b s)$

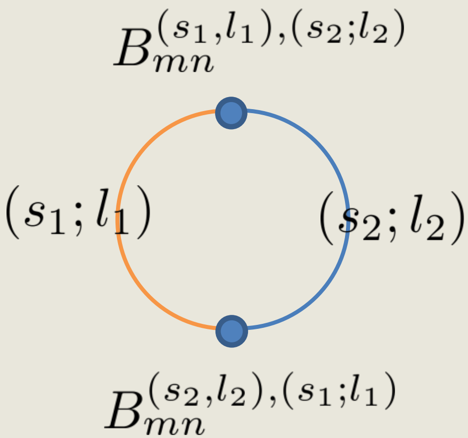
Matter side : Kac label

Boundary correlations on disk

1-point

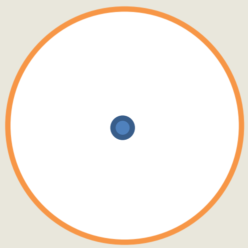


2-point

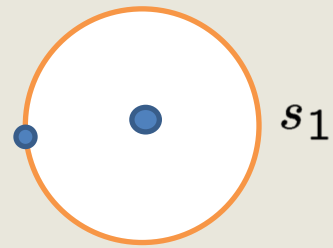


FZZ(2000),

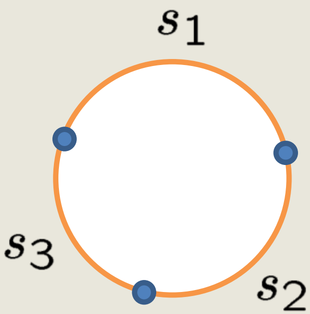
Bulk and Boundary correlations



Bulk correlation



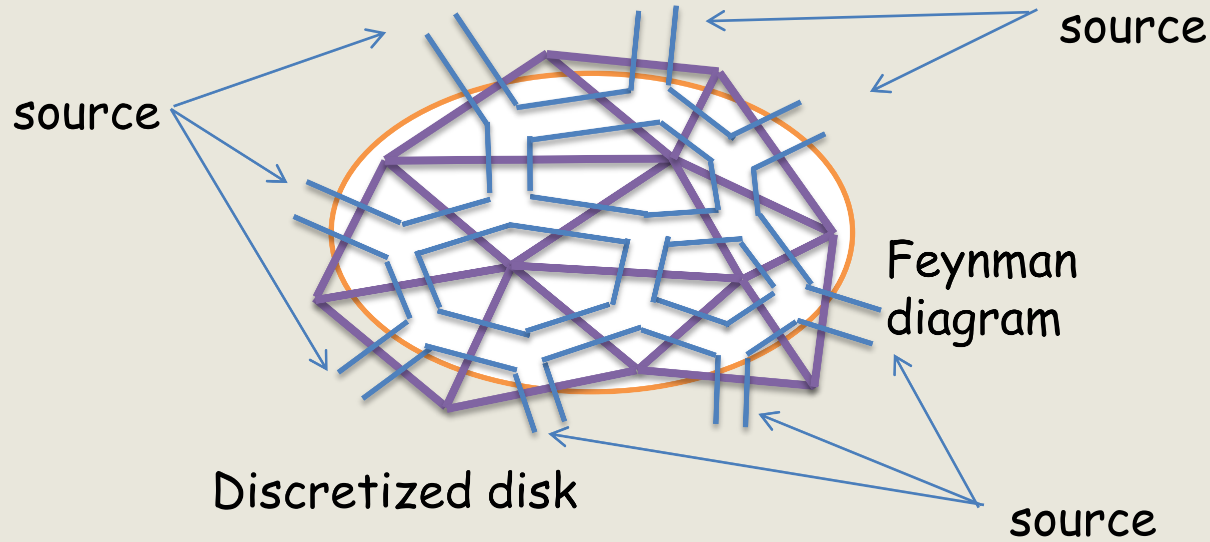
Bulk-boundary correlation



boundary correlation

FZZ(2000), Hosomichi(2008), Ponsot&Techner(2002)

What is the boundary (changing) operator in matrix model ?



- Large loop operator $\mathrm{tr}M^l$ is inserted in the path integral.

$$Z_{\text{disk}} := \int dM \left(\sum_l \frac{1}{l} x^{-l}\, \mathrm{tr} M^l \right) e^{-\mathrm{tr}V(M)} \sim -\langle \mathrm{tr} \log(x-M) \rangle$$

$$x = x_c + \sqrt{\mu}\cosh(\pi b s_1)$$ = Boundary cosmological constant

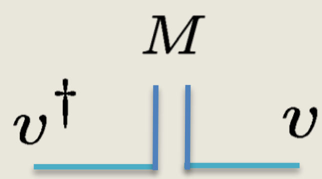
Boundary: introduce complex vectors

$$e^{Z} := \int dM dv dv^{\dagger}\, e^{-\mathrm{tr}V(M) - v^{\dagger}\cdot(x-M)\cdot v}$$

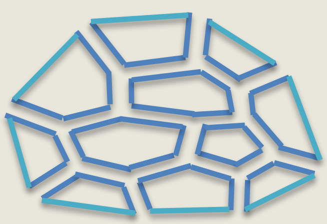
$$Z = \sum_{h=0}^{\infty} Z_h \qquad Z_{\text{disk}} = Z_1 = -\langle \mathrm{tr}\log(x-M)\rangle$$

Feynman rule for boundary

Propagator for vectors



Interaction for vectors



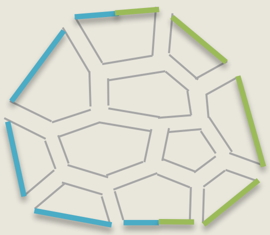
Vector loop forms the boundary.

Boundary changing : introduce 'flavors'

$$e^Z := \int dM \prod_{s=1}^{2} dv^{(s)} dv^{(s)\dagger}\, e^{-\mathrm{tr}V(M) - \sum_{s,t} v^{(s)\dagger}\cdot C^{(s,t)}(M)\cdot v^{(t)}}$$

$$C(M) = \begin{pmatrix} C^{(1,1)} & C^{(1,2)} \\ C^{(2,1)} & C^{(2,2)} \end{pmatrix} = \begin{pmatrix} x - M & c \\ c^* & y - M \end{pmatrix}$$

c, c^* generate boundary changing operators.



$$\sim \left.\frac{\partial^2 Z_{\text{disk}}}{\partial c \partial c^*}\right|_{c=c^*=0} = \left\langle \mathrm{tr} \frac{1}{x-M} \frac{1}{y-M} \right\rangle = \langle \mathcal{B}_{11}^{xy} \mathcal{B}_{11}^{yx} \rangle$$

4 Propagators

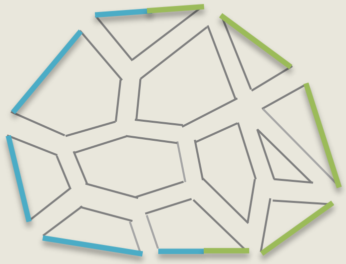
$v^{(1)\dagger}$ $v^{(1)}$ $\quad$ $v^{(2)\dagger}$ $v^{(2)}$ $\quad$ $v^{(2)\dagger}$ $v^{(1)}$ $\quad$ $v^{(1)\dagger}$ $v^{(2)}$

$\sim c^*$ $\quad$ $\sim c$

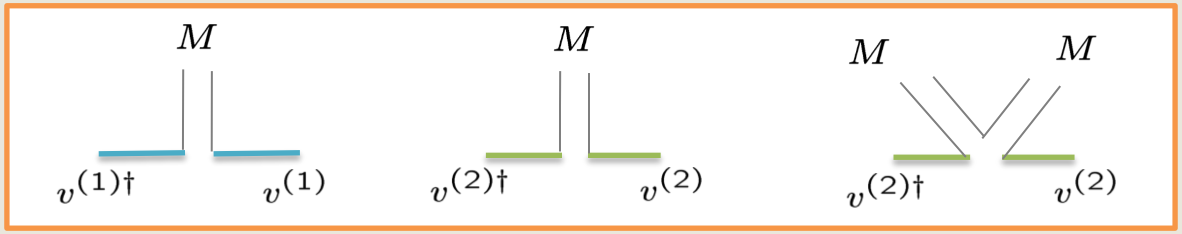
Nontrival interaction: introduce polynomials

$$e^Z := \int dM \prod_{s=1}^{2} dv^{(s)} dv^{(s)\dagger}\, e^{-\mathrm{tr}V(M) - \sum_{s,t} v^{(s)\dagger} \cdot C^{(s,t)}(M) \cdot v^{(t)}}$$

$$C(M) = \begin{pmatrix} C^{(1,1)} & C^{(1,2)} \\ C^{(2,1)} & C^{(2,2)} \end{pmatrix} = \begin{pmatrix} x - M & c \\ c^* & y_0 + y_1 M + M^2 \end{pmatrix}$$



$$\sim \frac{\partial^2 Z_{\text{disk}}}{\partial c \partial c^*}\bigg|_{c=c^*=0} = \left\langle \mathrm{tr} \frac{1}{x - M} \frac{1}{y_0 + y_1 M + M^2} \right\rangle = \langle \mathcal{B}_{12}^{xy_1} \mathcal{B}_{12}^{y_1 x} \rangle$$



$$Z_{\mathrm{disk}} = -\left\langle \mathrm{Tr}\log \begin{pmatrix} x - M & c \\ c^* & y_0 + y_1 M + M^2 \end{pmatrix} \right\rangle$$

(1) x and y_1 are source to the loop operator

$$\left.\frac{\partial Z_{MM}}{\partial x}\right|_{c=c^*=0} = \left\langle \frac{1}{x-M} \right\rangle \sim \cosh\left(\frac{\pi s_1}{b}\right)$$

$$x = x_c + \sqrt{\mu}\cosh(\pi b s_1)$$

$$\left.\frac{\partial Z_{MM}}{\partial y_1}\right|_{c=c^*=0} = \left\langle \frac{1}{y_0 + y_1 M + M^2} M \right\rangle \sim \cosh\left(\frac{\pi s_2}{b}\right)$$

$$Z_{\text{disk}} = -\left\langle \mathrm{Tr}\log \begin{pmatrix} x-M & c \\ c^* & y_0 + y_1 M + M^2 \end{pmatrix} \right\rangle$$

(2) y_0 is the source to $(1,3)$ boundary operator

$$\left.\frac{\partial Z_{\text{disk}}}{\partial y_0}\right|_{c=c^*=0} = \left\langle \mathrm{tr}\frac{1}{y_0 + y_1 M + M^2} \right\rangle = \frac{w(\tilde{y}_-) - w(\tilde{y}_+)}{\tilde{y}_+ - \tilde{y}_-}$$

$$w(\tilde{y}_\pm) := \left\langle \mathrm{tr}\frac{1}{\tilde{y}_\pm - M} \right\rangle \quad \left\{ \begin{array}{l} \tilde{y}_+ \tilde{y}_- = y_0 \\ \tilde{y}_+ + \tilde{y}_- = -y_1 \end{array} \right.$$

$\tilde{y}_\pm = \sqrt{\mu}\cosh(\pi b s_\pm)$ are determined due to CFT requirement

$$w(\tilde{y}_+) = w(\tilde{y}_-) \Longrightarrow \boxed{s_\pm = s_i \pm ib}$$

- l-th order polynomial, $Z_{\text{disk}} = -\langle \mathrm{Tr} \log C(M) \rangle$

$$C(M) = \begin{pmatrix} x - M & c \\ c^* & (\tilde{y}_1 - M)(\tilde{y}_2 - M)\cdots(\tilde{y}_l - M) \end{pmatrix}$$

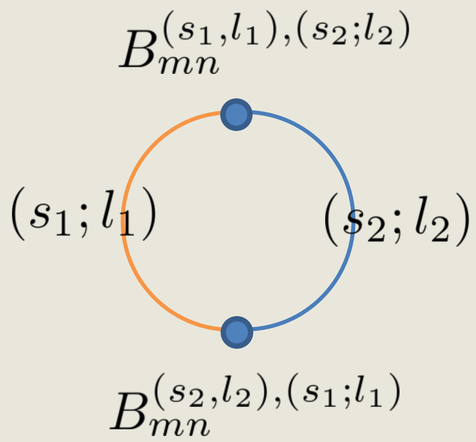
$$\boxed{s_{i+1} = s_i + 2ib} \quad \text{for} \quad \tilde{y}_i = \sqrt{\mu}\cosh(\pi b s_i)$$

- Non-trivial boundary = sum of (1,1) boundary with different s-values : leads to brane-decompostion

$$Z_{\rm disk}(s,(1,\ell)) = \sum_{k=-(\ell-1);2}^{\ell-1} Z_{\rm disk}(s+ikb,(1,1))$$

$$Z_{\rm annular}(s,(1,\ell)|s',(1,m)) = \sum_{\ell'=-(\ell-1);2}^{\ell-1}\ \sum_{m'=-(m-1);2}^{m-1} Z_{\rm annular}(s+i\ell' b,(1,1)|s'+im'b,(1,1))$$

Finding boundary and bulk correlations



2-point correlation

$$Z_{\text{disk}} = -\left\langle \text{Tr}\log\begin{pmatrix} x-M & c \\ c^* & (\tilde{y}_+ - M)(\tilde{y}_- - M) \end{pmatrix}\right\rangle$$

$$\left.\frac{\partial^2 Z_{\text{disk}}}{\partial c \partial c^*}\right|_{c=c^*=0} = \left\langle \text{tr}\frac{1}{x-M}\frac{1}{(\tilde{y}_+ - M)(\tilde{y}_- - M)}\right\rangle$$

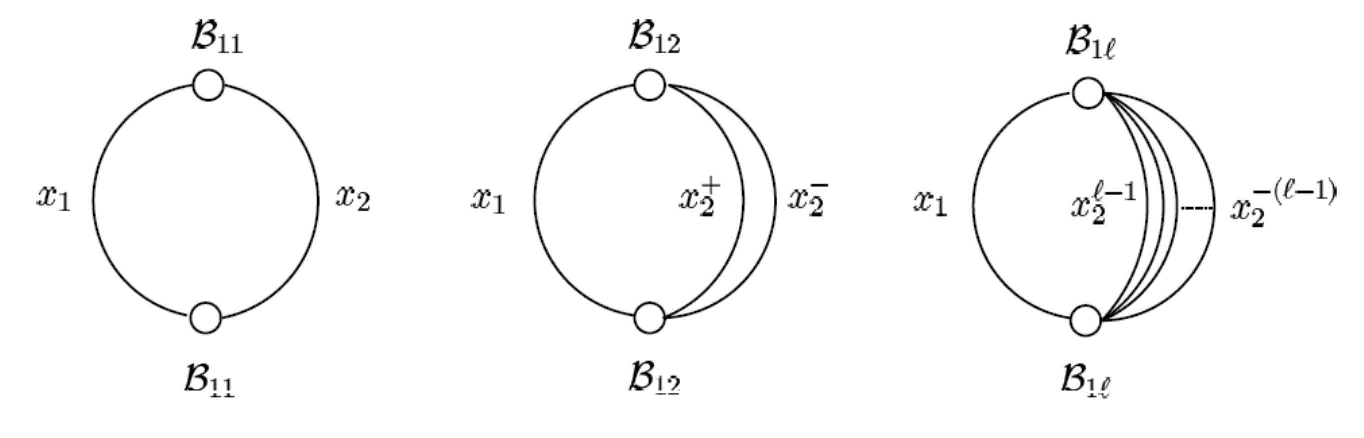
$$\sim \frac{u_0^{p-\frac{3}{2}}\cosh\left(\frac{\pi(s_1+s_2)}{2b}\right)\cosh\left(\frac{\pi(s_1+s_2)}{2b}\right)}{2\sinh\left(\frac{\pi b(s_1+s_2+ib)}{2}\right)\sinh\left(\frac{\pi b(s_1+s_2-ib)}{2}\right)\sinh\left(\frac{\pi b(s_1-s_2+ib)}{2}\right)\sinh\left(\frac{\pi b(s_1-s_2-ib)}{2}\right)}$$

$$\equiv \langle \mathcal{B}_{12}^{s_1 s_2}\mathcal{B}_{12}^{s_2 s_1}\rangle$$

- For l-th order interaction,

$$Z_{\mathsf{disk}} = -\left\langle \mathsf{Tr}\log\begin{pmatrix} x-M & c \\ c^* & \prod_{i=1}^{l}(\tilde{y}_i - M) \end{pmatrix}\right\rangle \quad \begin{cases} \tilde{y}_i = \sqrt{\mu}\cosh(\pi b s_i) \\ s_{i+1} = s_i + 2i\pi b \end{cases}$$

$$\left.\frac{\partial^2 Z_{\mathsf{disk}}}{\partial c \partial c^*}\right|_{c=c^*=0} \equiv \langle \mathcal{B}_{1l}^{s_1 s_2} \mathcal{B}_{1l}^{s_2 s_1} \rangle$$



◆ Generalization between two boundaries $s_1;(1,l)$ and $s_2;(1,m)$

$$Z_{\rm disk} = -\left\langle {\rm Tr}\,\log \begin{pmatrix} \prod_{i=1}^{\ell}(x_i - M) & c_0 \\ c_0^* & \prod_{i=1}^{m}(y_i - M) \end{pmatrix} \right\rangle$$

- The order of polynomials in the diagonal blocks

$\longrightarrow$ BC for matter. (1,l) & (1,m)

- Parameters x, y in the diagonal blocks

$\longrightarrow$ BC for gravity and sources to

Boundary operator $B_{1,\ell+m-1}$

- Generalizations: boundary operators $B_{1,l+m-1-2k}$

$k = 0, 1, \cdots, l+m-1$

$$Z_{\rm disk} = -\left\langle {\rm Tr}\,\log \begin{pmatrix} \prod_{i=1}^{\ell}(x_i - M) & c_0 + c_1\, p_1^{\ell,m}(M) + \cdots \\ c_0^* + c_1^*\, p_1^{\ell,m}(M)^* + \cdots & \prod_{i=1}^{m}(y_i - M) \end{pmatrix} \right\rangle$$

- c_k in the off-diagonal blocks : source to $B_{1,\ell+m-2k}$

- $p_k(M)$ = polynomial of order k :contains the complete information of boundary operator resonance

$$p_j^{(\ell,m)}(M) = \sum_{k=0}^{j} a_{jk}^{(\ell,m)}(x,y)\; M^k$$

◆ Explicit form of $p_k(M)$ (BIR: 1012.1467 (PLB 698))

$$\left\langle \mathrm{tr} \frac{1}{F_\ell(x,M)F_m(y,M)} P_j^{(\ell m)}(x,y,M) P_k^{(m\ell)}(x,y,M) \right\rangle \propto \delta_{jk}$$

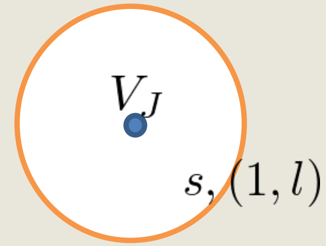
$$[k] \equiv U_{k-1}(\cos \pi b^2) = \frac{\sin(k\pi b^2)}{\sin(\pi b^2)}$$

$$P_0^{(\ell m)}(x,y,M) = 1$$

$$P_1^{(\ell m)}(x,y,M) = M - \frac{[m-1]x + [\ell-1]y}{[\ell+m-2]},$$

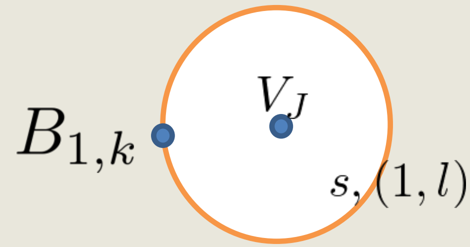
$$P_2^{(\ell m)}(x,y,M) = M^2 - \frac{[2]([m-2]x + [\ell-2]y)}{[\ell+m-4]} M - u_0^2 \sin^2 \pi b^2 \frac{[\ell-1][m-1]}{[\ell+m-3]} + \frac{[m-1][m-2]x^2 + [\ell-1][\ell-2]y^2 + [2][\ell-2][m-2]xy}{[\ell+m-3][\ell+m-4]}$$

- Bulk one point with $s, (1,l)$ boundary



One needs bulk resonance transformation from kdv frame to CFT frame, Belavin and Rim (1001.4356, PLB 687)

- Bulk-boundary correlation $\langle V_J B_{1,k} \rangle$



One needs bulk-boundary resonance transformation BIR (1107.4186)

$$t_a^{(\ell)} = p_a^{(l)}(\mu,\mu_B) + \sum_{k=a}^{l-2} p_a^{(l),k}(\mu,\mu_B)\lambda_k^{(l)} + \sum_{k,m=a}^{l-2} p_a^{(l),km}(\mu,\mu_B)\lambda_k^{(l)}\lambda_m^{(l)}$$
$$+ \sum_{J=p+1-l+a}^{p-2} q_a^{(l),J}(\mu,\mu_B)\lambda_J + \cdots$$

Summary and remarks

- Boundary description of matrix model is constructed including boundary correlation numbers and bulk-boundary correlation

- annulus amplitude is under investigation.

- Generalization to non-minimal theory is not achieved yet.