The nonlinear integral equation in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM.

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Based on:

arXiv: 0802.0027 with D.Bombardelli, D.Fioravanti;
arXiv: 0804.2893, 0808.1886, 0901.3161, 0911.2425, with D.Fioravanti, P.Grinza
arXiv: 0901.3147, with. G.Infusino, D.Fioravanti
arXiv: 1004.1081, with D.Fioravanti

Works in progress with G.Infusino, D.Fioravanti; D.Fioravanti.

Problem: Compute anomalous dimensions γ of linear combinations of twist L operators in planar $\mathcal{N}=4$ SYM ($\lambda = 8\pi^2 g^2$),

$$\mathcal{O}(x) = \sum_{k_1+\ldots+k_L=s} c_{k_1,\ldots,k_L} \mathrm{Tr}\left(\mathcal{D}_+^{k_1}\phi(x)\ldots\mathcal{D}_+^{k_L}\phi(x)\right),$$

which diagonalise the dilatation generator

$$\hat{D}\,\mathcal{O} = \Delta\,\mathcal{O} = (L+s+\gamma)\mathcal{O}, \quad \langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{const}{|x-y|^{2\Delta}}.$$

(Some) Motivations:

- Test of AdS/CFT [1998, Maldacena; Gubser, Klebanov, Polyakov; Witten]: Δ dual to energy of string states

- For twist $L=2$, anomalous dimensions are Mellin transforms of splitting kernels of DGLAP equations

$$\gamma_{ab}(s) = \int_0^1 dx\; x^{s-1}\, W_{b\to a}(x)$$

Integrability in planar limit ($N_c \to \infty$)

- QCD, one loop: scattering of reggeised gluons [Lipatov;Korchemsky, Faddeev] and aligned-helicity twist operators [Lipatov;Korchemsky, Braun, Belitsky, Derkachov, Manashov]
- $\mathcal{N}$=4 SYM scalar sector, one loop: Dilatation operator= XXX Hamiltonian [2002, Minahan, Zarembo]
- $\mathcal{N}$=4 SYM 'full' theory [2003-2006, Beisert, Kristjansen, Staudacher....] 'Bethe' ansatz equations (no clear physical model, spin chain-like)
- Progress for AdS/CFT: gauge/string bridge: **asymptotic** Bethe ansatz equations with string dressing [2003-2006,Arutyunov,Frolov,Staudacher;Beisert, Eden, Staudacher], perturbatively reliable up to $O(g^{2L-2})$ (wrapping problem)
- Final answer: From Bethe ansatz equations to TBA or Y system+relations [2009, Bombardelli, Cavaglià,Fioravanti,Tateo; Gromov, Kazakov,Vieira; Arutyunov,Frolov]; towards nonlinear integral equations [2011, Balog, Hegedus; Gromov, Kazakov, Leurent, Tsuboi,Volin]

For twist operators $\mathcal{O}(x)$, how does integrability work?
Operator dimension $\Delta = L + s + \gamma$, with

$$\gamma = \frac{ig^2}{2} \sum_{k=1}^{s} \left[\left(\frac{1}{x^+(u_k)} \right) - \left(\frac{1}{x^-(u_k)} \right) \right]$$

u_k are solutions of 'Bethe' equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L \left(\frac{1 + \frac{g^2}{2x_k^{-2}}}{1 + \frac{g^2}{2x_k^{+2}}} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{s} \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^+}} \right)^2 e^{2i\theta(u_k, u_j)}$$

where $x_k^{\pm} = x(u_k \pm i/2)$, $x(u) = \frac{u}{2}\left[1 + \sqrt{1 - \frac{2g^2}{u^2}}\right]$ and $\theta(u, v)$ is the dressing phase.

Caveat: Wrapping problem: in perturbative expansion Bethe equations are reliable up to $O(g^{2L-2})$. TBA should be used...

BUT: Simplification in the $s \to \infty$ limit! Wrapping problem affect $O\left(\frac{(\ln s)^2}{s^2}\right)$ terms [2008-2009, Bajnok, Janik, Lukowski, Rej, Velizhanin]

In general Bethe ansatz equations are equivalently written as Non Linear Integral Equations $\Longrightarrow$ Helps for studying configurations with large numbers s of Bethe roots.

Counting function: $Z(u) = \Phi(u) - \sum_{k=1}^{s} \phi(u, u_k)$, where

$$\begin{aligned} \Phi(u) &= -2L \arctan 2u - iL \ln \left(\frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right), \\ \phi(u, v) &= 2 \arctan(u - v) - 2i \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right]. \end{aligned}$$

Bethe roots $u_k \in [-b, b]$ are real $\Longrightarrow Z(u_k) = \pi(2\mathbb{Z} + 1 + L)$.
IMPORTANT: also L real 'holes' x_h are present: $Z(x_h) = \pi(2\mathbb{Z} + 1 + L)$.
$x_1, ..., x_{L-2} \in [-b, b]$, $x_{L-1} = -x_L > b$.

Using Cauchy theorem [1996, Fioravanti, Mariottini, Quattrini, Ravanini]

$$\sum_{k=1}^{s} O(u_k) + \sum_{h=1}^{L} O(x_h) = -\int_{-\infty}^{+\infty} \frac{dv}{2\pi} O(v) Z'(v) +$$
$$+ \int_{-\infty}^{+\infty} \frac{dv}{\pi} O(v) \frac{d}{dv} \mathrm{Im} \ln[1 + (-1)^L e^{iZ(v-i0^+)}] \, .$$

we write the Non Linear Integral Equation for the counting function

$$Z(u) = F(u) + 2 \int_{-\infty}^{+\infty} dv G(u,v) \mathrm{Im} \ln[1 + (-1)^L e^{iZ(v-i0^+)}]$$

where $F(u)$ and $G(u,v)$ satisfy the linear integral equations

$$F(u) = \Phi(u) + \sum_{h=1}^{L} \phi(u, u_h) - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \frac{d}{dv} \phi(u,v) F(v)$$

$$G(u,v) = \frac{1}{2\pi} \frac{d}{dv} \phi(u,v) - \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \frac{d}{dw} \phi(u,w) G(w,v)$$

Eigenvalues of observables:

$$\sum_{k=1}^{s} O(u_k) = \int_{-\infty}^{+\infty} \frac{dv}{2\pi} O^{'}(v) F(v) - \sum_{h=1}^{L} O(x_h) +$$
$$+ \quad 2\int_{-\infty}^{+\infty} \frac{dv}{2\pi} O^{'}(v) \int_{-\infty}^{+\infty} dw [G(v,w) - \delta(v-w)] \mathrm{Im} \ln \left[1 + e^{iZ(w-i0^+)}\right] .$$

Momentum:

$$O(u) = \frac{\Phi(u)}{iL}$$

Anomalous dimension:

$$O(u) = \frac{ig^2}{2} \left[\left(\frac{1}{x^+(u)} \right) - \left(\frac{1}{x^-(u)} \right) \right]$$

Technical problem: 'kernel' $\phi(u,v)$ is not a function only of $u-v$.
BUT: $\phi(u,v) = 2\arctan(u-v) + \phi_H(u,v)$
$\phi_H(u,v)$ is (perturbatively) 'separable': in Fourier space:

$$\begin{aligned}
\hat{\phi}_H(k,t) &= -8i\pi^2 \frac{e^{-\frac{|t|+|k|}{2}}}{k|t|} \Bigg[\sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}gk) J_r(\sqrt{2}gt) \frac{1-\mathrm{sgn}(kt)}{2} + \\
&+ \mathrm{sgn}(t) \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g)(-1)^{r+\nu} \Big(J_{r-1}(\sqrt{2}gk) J_{r+2\nu}(\sqrt{2}gt) - \\
&- J_{r-1}(\sqrt{2}gt) J_{r+2\nu}(\sqrt{2}gk) \Big) \Bigg] .
\end{aligned}$$

In Fourier space Neumann expansion on a basis of Bessel functions is convenient.

Large s (high spin) simplifications:

- Non linear term:

$$2\int_{-\infty}^{+\infty} dvG(u,v)\mathrm{Im}\ln[1+(-1)^L\, e^{iZ(v-i0^+)}] = -2u\ln 2 + O\left(\frac{u^3}{s^2}\right)$$

- External, large holes: $x_{L-1} = -x_L = \frac{s}{\sqrt{2}}\left(1+\frac{L-1+\gamma}{s}+O\left(\frac{1}{s^2}\right)\right)$

Then $Z(u) = F(u) - 2u\ln 2 + O\left(\frac{u^3}{s^2}\right)$.
Equation for $F(u)$

$$F(u) = \Phi(u) + \sum_{h=1}^{L}\phi(u,u_h) - \int_{-\infty}^{+\infty}\frac{dv}{2\pi}\frac{d}{dv}\phi(u,v)F(v)$$

is driven by the term containing external holes.

For large s

$$Z(u) = Z^{(-1)}(u)\ \ln s + Z^{(0)}(u) + \sum_{n=1}^{\infty} \frac{Z^{(n)}(u)}{(\ln s)^n} + ...$$

i.e. $\ln s$ is the scale
Anomalous dimension:

$$\gamma = f(g) \ln s + f_{sl}(g, L) + \sum_{n=1}^{\infty} \frac{\gamma^{(n)}(g, L)}{(\ln s)^n} + ...$$

$f(g)$:(twice) cusp anomalous dimension of Wilson loops

Internal holes $x_1, ..., x_{L-2}$:

$$Z(x_h) = \pi(2\mathbb{Z} + 1 + L)$$

'Dynamics' of the internal holes gives different states.
From now on we study the minimal anomalous dimension state:

$$Z(x_h) = \pi(2h + 1 - L), \quad h = 1, ..., L-2, \Rightarrow x_h = \sum_{n=1}^{\infty} \frac{x_h^{(n)}}{(\ln s)^n} + ...$$

It is useful to define

$$
\begin{aligned}
S(k) \quad &= \quad \frac{\sinh \frac{|k|}{2}}{\pi |k|} \left\{ ik \left[\hat{Z}(k) + \frac{2\hat{L}(k)}{e^{|k|}-1} \right] + \frac{\pi L}{\sinh \frac{|k|}{2}} \left(1 - e^{-\frac{|k|}{2}} \right) - \right. \\
&- \quad \left. \frac{2\pi e^{-|k|}}{1-e^{-|k|}} \sum_{h=1}^{L} [\cos k x_h - 1] \right\} ,
\end{aligned}
$$

$$
\hat{L}(k) = \int_{-\infty}^{+\infty} du e^{-iku} \mathrm{Im} \ln[1 + (-1)^L \, e^{iZ(u-i0^+)}]
$$

Then, Neumann expansion

$$
S(k) = \sum_{p=1}^{\infty} S_p(g) \frac{J_p(\sqrt{2}gk)}{k}
$$

Linear integral equation for $S(k)$ translates into an infinite linear system of equations for $S_p(g)$.
IMPORTANT: anomalous dimension $\gamma = 2S(0) = \sqrt{2}gS_1(g)$.

$S(k)$ expands in (inverse) powers of $\ln s$

At leading order $\ln s$ one has the BES equation [2006, Beisert, Eden, Staudacher]. In terms of coefficients:

$$S_{2p}^{(BES)}(g) = -4p \sum_{m=1}^{\infty} Z_{2p,m}(g)(-1)^m S_m^{(BES)}(g)\,,$$

$$S_{2p-1}^{(BES)}(g) = 2\sqrt{2}g\,\delta_{p,1} - 2(2p-1)\sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m^{(BES)}(g)\,,$$

where we introduced the notation

$$Z_{n,m}(g) = \int_0^{+\infty} \frac{dt}{t} \frac{J_n(\sqrt{2}gt)J_m(\sqrt{2}gt)}{e^t - 1}\,.$$

REMARK: $(-1)^m$ is the only effect of 'dressing' phase

$\sqrt{2}gS_1^{(BES)}(g) = f(g)$

At order $(\ln s)^0$ one has [2009, Freyhult, Zieme], [0802.0027,0901.3147,0901.3161]
$S_p^{(0)}(g) = S_p^{extra}(g) + (L-2) S_p^{(-1,1)}(g)$, where

$$
\begin{aligned}
S_{2p}^{extra}(g) &= 4 + 8p \int_0^{+\infty} \frac{dt}{t} \frac{J_{2p}(\sqrt{2}gt)}{e^t - 1} + \\
&- 4p \sum_{m=1}^{\infty} Z_{2p,m}(g)(-1)^m S_m^{extra}(g)\,, \\
S_{2p-1}^{extra}(g) &= 2\sqrt{2} g \gamma_E \delta_{p,1} + 4(2p-1) \int_0^{+\infty} \frac{dt}{t} \frac{\tilde{J}_{2p-1}(\sqrt{2}gt)}{e^t - 1} - \\
&- 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m^{extra}(g)\,,
\end{aligned}
$$

where $\tilde{J}_{2p-1}(x) = J_{2p-1}(x) - \delta_{p,1} \frac{x}{2}$.

... and

$$
\begin{aligned}
S_{2p}^{(-1,1)}(g) &= 2 - 4p \int_0^{+\infty} \frac{dt}{t} \frac{J_{2p}(\sqrt{2}gt)}{1+e^{\frac{t}{2}}} - \\
&- 4p \sum_{m=1}^{\infty} Z_{2p,m}(g)(-1)^m S_m^{(-1,1)}(g) , \\
S_{2p-1}^{(-1,1)}(g) &= -2(2p-1) \int_0^{+\infty} \frac{dt}{t} \frac{J_{2p}(\sqrt{2}gt)}{1+e^{\frac{t}{2}}} - \\
&- 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m^{(-1,1)}(g) .
\end{aligned}
$$

We get $f_{sl}(g, L) = \sqrt{2} g S_1^{(0)}(g)$.

At orders $(\ln s)^{-n}$ [0911.2425]

$$
\begin{aligned}
S_{2p}^{(n)}(g) &= -2p\int_0^{+\infty}\frac{dt}{t}\frac{J_{2p}(\sqrt{2}gt)}{\sinh\frac{t}{2}}\sum_{h=1}^{L-2}(\cos tx_h-1)- \\
&- 4p\sum_{m=1}^{\infty}Z_{2p,m}(g)(-1)^m S_m^{(n)}(g)\,, \\
S_{2p-1}^{(n)}(g) &= -(2p-1)\int_0^{+\infty}\frac{dt}{t}\frac{J_{2p-1}(\sqrt{2}gt)}{\sinh\frac{t}{2}}\sum_{h=1}^{L-2}(\cos tx_h-1)- \\
&- 2(2p-1)\sum_{m=1}^{\infty}Z_{2p-1,m}(g)S_m^{(n)}(g)\,.
\end{aligned}
$$

$$\gamma^{(n)}(g,L)=\sqrt{2}gS_1^{(n)}(g)$$

One easily gets weak coupling perturbative expansions (for $S_p(g)$ and anomalous dimension $\gamma = \sqrt{2} g S_1(g)$)[0802.0027, 0901.3161, 0911.2425]

BUT also strong coupling asymptotic expansions have been obtained! [0804.2893, 0808.1866]. For instance:

$$
\begin{aligned}
S_{2p-1}^{(-1,1)}(g) &\doteq (2p-1)\sum_{n'=1}^{p}(-1)^{n'}\frac{\Gamma(p+n'-1)}{\Gamma(p-n'+1)}\frac{b_{2n'-1}}{g^{2n'-1}}\,, \\
S_{2p}^{(-1,1)}(g) &\doteq -2p\sum_{n'=1}^{p}(-1)^{n'}\frac{\Gamma(p+n')}{\Gamma(p-n'+1)}\frac{b_{2n'}}{g^{2n'}}\,,
\end{aligned}
$$

where the coefficients $b_{n'}$ sum up to the generating function

$$
\sum_{n'=0}^{\infty} b_{n'} t^{n'} = \frac{1}{\cos\frac{t}{\sqrt{2}} - \sin\frac{t}{\sqrt{2}}}\,.
$$

Similar expansions for $S_p^{(n)}(g)$.

A scaling limit [2006, Belitsky, Gorsky, Korchemsky; 2007, Alday, Maldacena; Freyhult, Rej, Staudacher]

$$s \to \infty\,, \quad L \to \infty\,, \quad j = \frac{L-2}{\ln s} \quad \text{fixed}\,.$$

In this limit:

$$\gamma = \ln s \sum_{r=0}^{\infty} f_r(g) j^r + \sum_{n=0}^{\infty} (\ln s)^{-n} \sum_{r=0}^{\infty} f_r^{(n)}(g) j^r + \ldots$$

$f_0(g) = f(g)$, $f_1(g) = \sqrt{2} g S_1^{(-1,1)}(g)$
In general, $f_r^{(n)}(g)$ depend on internal holes.

Internal holes becomes infinite...they fill the interval $[-c, c] \subset [-b, b]$.

$$\sum_{h=1}^{L-2} (\cos tx_h - 1) =$$

$$= \quad -\int_{-c}^{c} \frac{dv}{2\pi}(\cos tv - 1)\frac{d}{dv}\left\{Z(v) - 2\mathrm{Im}\ln[1 + (-1)^L e^{iZ(v-i0^+)}]\right\}$$

The nonlinear term is evaluated:

$$\int_{-c}^{c} \frac{dv}{\pi}(\cos tv - 1)\frac{d}{dv}\mathrm{Im}\ln[1 + (-1)^L e^{iZ(v-i0^+)}] = -\frac{\pi}{6}\frac{t\sin tc}{Z'(c)} + ...$$

and c comes from $Z(c) = -2\pi j \ln s$.

Working out non linear terms coming from internal holes, the various $f_r^{(n)}(g)$ are obtained [0911.2425,1004.1081]

CONCLUSIONS

- **We studied non linear integral equations equivalent to Bethe equations for $sl(2)$ sector of planar $\mathcal{N} = 4$ SYM**
- **For large number of Bethe roots, weak coupling and strong coupling expansions were provided; Non linear terms were computed**
- **Nonlinear integral equations are becoming more and more important in AdS/CFT**
- **Learning about techniques which solve them can be useful**