Exactly Solvable Quantum Mechanics and Infinite Families of Multi-indexed Orthogonal Polynomials

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Outline

Discovery of ∞-many Multi-Indexed Polynomials

- Infinitely many *orthogonal polynomials satisfying second order differential equations*, after Hermite, Laguerre and Jacobi polynomials, (Odake-Sasaki, 2009-11)
- called Infinite Families of Multi-Indexed orthogonal polynomials $P_{\mathcal{D},n}(x)$, $\mathcal{D}=\{d_1,\ldots,d_M\}$, $d_j\in\mathbb{N}$:

$$\int P_{\mathcal{D},n}(x)P_{\mathcal{D},m}(x)\mathcal{W}_{\mathcal{D}}(x)dx=h_{\mathcal{D},n}\delta_{n\,m}$$

- Solutions of *exactly solvable Schrödinger eq.*
- degree $\ell+n$ polynomial in x, but forming a complete set,
- No three term recurrence relations
- *global solutions* of Fuchsian differential equations with $3+\ell$ *regular singularities*, utterly unknown for $\ell>1$ before 2009

∞-many Exceptional Orthogonal Polynomials

- for $\mathcal{D} = \{\ell\}$, $\ell \geq 1 \Rightarrow$ Exceptional orthogonal (Jacobi & Laguerre, 2009) polynomials $\ell = 1, 2, \ldots,$, $P_{\ell,n}(x)$, $n = 0, 1, 2, \ldots,$, (n counts nodes):
$$\int P_{\ell,n}(x) P_{\ell,m}(x) \mathcal{W}_\ell(x) dx = h_{\ell,n} \delta_{n\,m}$$
- generalisation: *Exceptional (X_ℓ) Wilson, Askey-Wilson, Meixner-Pollaczek, continuous Hahn, Racah, q-Racah, dual Hahn, dual q-Hahn, little q-Jacobi, Meixner polynomials* are also discovered as solutions of *exactly solvable difference Schrödinger eq.* (discrete quantum mechanics) (Odake-Sasaki, '09, '10, '11)

History

- $\ell = 1$ exceptional Laguerre and Jacobi polynomials, introduced by Gómez-Ullate et al (July '08) within Sturm-Liouville theory, by avoiding Bochner's theorem
- quantum mechanical reformulation with shape-invariant potentials by Quesne (July '08)
- $\forall \ell \geq 1$ exceptional Laguerre and Jacobi polynomials discovered by Odake-Sasaki (June '09)
- new (type II) $\ell = 2$ exceptional Laguerre & Jacobi introduced by Quesne (June '09)
- $\forall \ell \geq 2$ type II exceptional Laguerre & Jacobi introduced by Odake-Sasaki (Nov. '09)
- precursor of X_ℓ Laguerre by Junker-Roy ('97)

General Structure of Factorised Hamiltonians

Problem: Find the complete set of eigenvalues and eigenfunctions

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \quad \int \phi_n^2(x)dx < \infty, \quad n = 0, 1, 2, \ldots,$$

by adjusting the const. of $\mathcal{H} \Rightarrow \mathcal{E}_0 = 0$
$\Rightarrow$ Positive Semi-Definite Hamiltonian $\mathcal{H}$ (Hermitian Matrix)

$$0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots, \quad \Rightarrow \quad \mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$$

$$\mathcal{A} = d/dx - dw(x)/dx, \quad \mathcal{A}^\dagger = -d/dx - dw(x)/dx,$$

$\phi_0(x)$: groundstate wavefunction, no node, square integrable
$\phi_0(x) = e^{w(x)}$, $\quad w(x) \in \mathbb{R}$: prepotential $\mathcal{A}\phi_0(x) = 0$

$$\mathcal{H} = -d^2/dx^2 + V(x), \qquad V(x) = (dw(x)/dx)^2 + d^2w(x)/dx^2$$

Structure of Associated Hamiltonians

associated Hamiltonian $\mathcal{H}^{[1]} \stackrel{\text{def}}{=} \mathcal{A}\mathcal{A}^\dagger$ intertwining relation

$$\mathcal{A}\mathcal{A}^\dagger\mathcal{A} = \mathcal{A}\mathcal{H} = \mathcal{H}^{[1]}\mathcal{A}, \qquad \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger = \mathcal{A}^\dagger\mathcal{H}^{[1]} = \mathcal{H}\mathcal{A}^\dagger$$

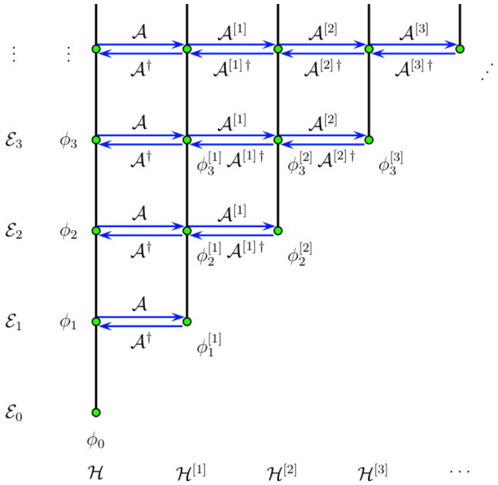
essentially iso-spectral

$$\phi_n^{[1]}(x) \stackrel{\text{def}}{=} \mathcal{A}\phi_n(x), \quad \phi_n(x) = \mathcal{A}^\dagger/\mathcal{E}_n\phi_n^{[1]}(x), \qquad n = 1, 2, \ldots,$$

$$\Rightarrow \mathcal{H}^{[1]}\phi_n^{[1]}(x) = \mathcal{E}_n\phi_n^{[1]}(x), \quad n = 1, 2, \ldots$$

removing the groundstate energy $\mathcal{E}_1$ from $\mathcal{H}^{[1]}$
$\Rightarrow$ positive semi-definite
$\Rightarrow$ factorisable $\mathcal{H}^{[1]} = \mathcal{A}^{[1]\dagger}\mathcal{A}^{[1]} + \mathcal{E}_1$ repeat!!

Schematic Picture of Crum's Theorem '55



Eigenfunctions etc for Crum's Theorem '55

in terms of Wronskian

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x) = \mathcal{E}_n\phi_n^{[M]}(x) \quad (n = M, M+1, \ldots),$$

$$\phi_n^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\phi_0, \phi_1, \ldots, \phi_{M-1}, \phi_n](x)}{\mathrm{W}[\phi_0, \phi_1, \ldots, \phi_{M-1}](x)},$$

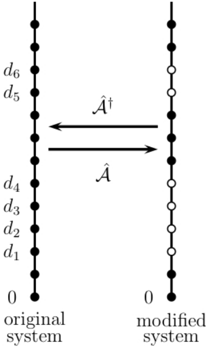
$$(\phi_m^{[M]}, \phi_n^{[M]}) = \prod_{j=0}^{M-1}(\mathcal{E}_n - \mathcal{E}_j) \cdot h_n\delta_{m\,n},$$

$$U^{[M]}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log\left|(\mathrm{W}[\phi_0, \phi_1, \ldots, \phi_{M-1}](x)\right|,$$

also called M-step Darboux-Crum transformation

Schematic Picture of Modified Crum's Theorem

delete eigenstates $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \ldots, d_M\}$, $d_j \geq 0$, $\prod_{j=1}^{M}(m - d_j) \geq 0$, $\forall m \in \mathbb{Z}_{\geq 0}$



$\hat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{A}_{d_1 \ldots d_\ell} \cdots \mathcal{A}_{d_1 d_2} \mathcal{A}_{d_1}$

Eigenfunctions etc for Modified Crum's Theorem

by Krein-Adler ('57, '94)

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x) = \mathcal{E}_n\phi_n^{[M]}(x) \quad (n \in \mathbb{Z}_{\geq 0}\backslash\mathcal{D}),$$

$$\phi_n^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}, \phi_n](x)}{\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x)},$$

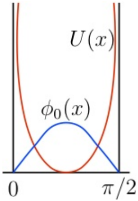
$$(\phi_m^{[M]}, \phi_n^{[M]}) = \prod_{j=1}^{M}(\mathcal{E}_n - \mathcal{E}_{d_j}) \cdot h_n\delta_{m\,n},$$

$$U^{[M]}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log\bigl|\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x)\bigr|$$

exactly solvable $\Rightarrow$ exactly solvable

Example: Pöschl-Teller potential $\Rightarrow$ Jacobi Polynomial

- $\mathcal{H} = -\dfrac{d^2}{dx^2} + U(x)$, $U(x) = \dfrac{g(g-1)}{\sin^2 x} + \dfrac{h(h-1)}{\cos^2 x} - (g+h)^2$, regular sing. $x = 0$, $g, 1-g$, $x = \pi/2$, $h, 1-h$, $\boldsymbol{\lambda} = \{g, h\}$,
- groundstate wavefunct. $\phi_0(x) = (\sin x)^g (\cos x)^h$, $\quad g, h > 0$, $w(x; \boldsymbol{\lambda}) = g \log \sin x + h \log \cos x$, $\quad 0 < x < \pi/2$,
- $\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(n+g+h)$, $\qquad \eta(x) \stackrel{\text{def}}{=} \cos 2x$
- $\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x) P_n^{(g-1/2, h-1/2)}(\eta(x))$, $\quad P_n$: Jacobi polynomial



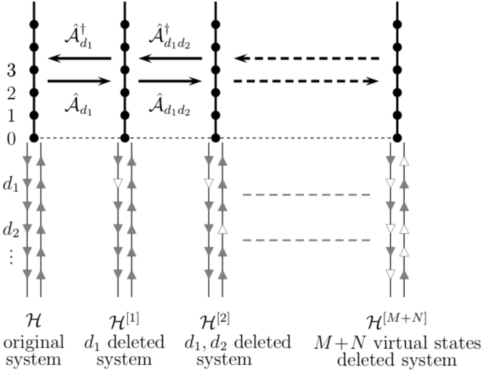
Multi-Indexed Orthogonal Polynomials 1

- Pöschl-Teller potential has virtual state solutions, type I and II, generated by the discrete symmetry of the potentail: $g \to 1-g$, and/or $h \to 1-h$
- negative energy and non-square integrable

$$\mathcal{H}\tilde{\phi}_{\rm v}(x) = \tilde{\mathcal{E}}_{\rm v}\tilde{\phi}_{\rm v}(x), \quad \tilde{\mathcal{E}}_{\rm v} < 0 \quad (\tilde{\phi}_{\rm v}, \tilde{\phi}_{\rm v}) = (1/\tilde{\phi}_{\rm v}, 1/\tilde{\phi}_{\rm v}) = \infty$$

- they have no zeros in $x \in (0, \pi/2)$
- delete virtual states à la Adler $\mathcal{D} \stackrel{\text{def}}{=} \{d_1^{\rm I}, \ldots, d_M^{\rm I}, d_1^{\rm II}, \ldots, d_N^{\rm II}\}$, $d_j^{\rm I,II} \geq 1$

Schematic Picture of Virtual States Deletion



Eigenfunctions etc after Virtual States Deletion

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x) = \mathcal{E}_n\phi_n^{[M]}(x) \quad (n \in \mathbb{Z}_{\geq 0}),$$

$$\mathcal{H}^{[M]}\tilde{\phi}_{\mathrm{v}}^{[M]}(x) = \tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\phi}_{\mathrm{v}}^{[M]}(x) \quad (\mathrm{v} \in \mathcal{V}\backslash\mathcal{D}),$$

$$\phi_n^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \phi_n](x)}{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)},$$

$$(\phi_m^{[M]}, \phi_n^{[M]}) = \prod_{j=1}^{M}(\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}) \cdot h_n\delta_{m\,n},$$

$$\tilde{\phi}_{\mathrm{v}}^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \tilde{\phi}_{\mathrm{v}}](x)}{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)},$$

$$U^{[M]}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log\left|\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)\right|.$$

shape inv. exactly solvable $\Rightarrow$ shape inv. exactly solvable

Multi-Indexed Orthogonal Polynomials 2

- explicit forms of type I virtual states

$$\tilde{\phi}^{\mathrm{I}}_{\mathrm{v}}(x) \stackrel{\text{def}}{=} (\sin x)^{g}(\cos x)^{1-h}\xi^{\mathrm{I}}_{\mathrm{v}}(\eta(x);g,h),$$
$$\xi^{\mathrm{I}}_{\mathrm{v}}(\eta;g,h) \stackrel{\text{def}}{=} P_{\mathrm{v}}(\eta;g,1-h),\quad \mathrm{v}=0,1,\ldots,[h-\tfrac{1}{2}]',$$
$$\tilde{\mathcal{E}}^{\mathrm{I}}_{\mathrm{v}} \stackrel{\text{def}}{=} -4(g+\mathrm{v}+\tfrac{1}{2})(h-\mathrm{v}-\tfrac{1}{2}),\quad \tilde{\boldsymbol{\delta}}^{\mathrm{I}} \stackrel{\text{def}}{=} (-1,1)$$

- explicit forms of type II virtual states

$$\tilde{\phi}^{\mathrm{II}}_{\mathrm{v}}(x) \stackrel{\text{def}}{=} (\sin x)^{1-g}(\cos x)^{h}\xi^{\mathrm{II}}_{\mathrm{v}}(\eta(x);g,h),$$
$$\xi^{\mathrm{II}}_{\mathrm{v}}(\eta;g,h) \stackrel{\text{def}}{=} P_{\mathrm{v}}(\eta;1-g,h),\quad \mathrm{v}=0,1,\ldots,[g-\tfrac{1}{2}]',$$
$$\tilde{\mathcal{E}}^{\mathrm{II}}_{\mathrm{v}} \stackrel{\text{def}}{=} -4(g-\mathrm{v}-\tfrac{1}{2})(h+\mathrm{v}+\tfrac{1}{2}),\quad \tilde{\boldsymbol{\delta}}^{\mathrm{II}} \stackrel{\text{def}}{=} (1,-1)$$

Multi-Indexed Orthogonal Polynomials 3

- Multi-Indexed Orthogonal Polynomials $P_{\mathcal{D},n}(\eta)$:

$$\phi_n^{[M,N]}(x) \equiv \phi_{\mathcal{D},n}(x;\boldsymbol{\lambda}) = (-4)^{M+N}\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta(x);\boldsymbol{\lambda}),$$

$$\psi_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x;\boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta(x);\boldsymbol{\lambda})},$$

- $\boldsymbol{\lambda}^{[M,N]} = (g+M-N, h-M+N)$,
  $\Xi_{\mathcal{D}}(\eta)$ has no node in $-1<\eta<1$;
- orthogonality

$$\int_{-1}^{1} d\eta\, \frac{W(\eta;\boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})^2} P_{\mathcal{D},m}(\eta;\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) = h_{\mathcal{D},n}\delta_{nm}$$

- Pearson's equation satisfied

Multi-Indexed Orthogonal Polynomials 4

- Explicit Forms

$$P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathrm{W}[\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_N,P_n](\eta) \times \Bigl(\frac{1-\eta}{2}\Bigr)^{(M+g+\frac{1}{2})N}\Bigl(\frac{1+\eta}{2}\Bigr)^{(N+h+\frac{1}{2})M}$$

$$\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathrm{W}[\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_N](\eta) \times \Bigl(\frac{1-\eta}{2}\Bigr)^{(M+g-\frac{1}{2})N}\Bigl(\frac{1+\eta}{2}\Bigr)^{(N+h-\frac{1}{2})M}$$

$$\mu_j = \Bigl(\frac{1+\eta}{2}\Bigr)^{\frac{1}{2}-h}\xi^{\mathrm{I}}_{d^{\mathrm{I}}_j}(\eta;g,h), \quad \nu_j = \Bigl(\frac{1-\eta}{2}\Bigr)^{\frac{1}{2}-g}\xi^{\mathrm{II}}_{d^{\mathrm{II}}_j}(\eta;g,h)$$

- $P_{\mathcal{D},n}(\eta)$ degree $\ell+n$, $\Xi_{\mathcal{D}}(\eta)$ degree ℓ;

$$\ell = \sum_{j=1}^{M} d^{\mathrm{I}}_j + \sum_{j=1}^{N} d^{\mathrm{II}}_j - \frac{1}{2}M(M-1) - \frac{1}{2}N(N-1) + MN \geq 1$$

Multi-Indexed Orthogonal Polynomials 4

- $P_{\mathcal{D},n}(\eta)$: global solution of a Fuchsian differential equation with $3+\ell$ regular singularities
- characteristic exponents $\rho = 0, 3$
- for $\mathcal{D} = \{\ell^I\}$ or $\{\ell^{II}\}$ $\Rightarrow$ the Exceptional Orthogonal Polynomials (derived in '09 by Odake-Sasaki)

Summary and Outlook

- Infinitely many new orthogonal polynomials satisfying second order differential or difference equations are discovered recently.
- Various concepts and methods of QM have much wider currency and utility in the theory of ordinary differential and difference equations than is usually regarded.
- Various properties of the Askey-scheme of hypergeometric orthogonal polynomials can be understood in a unified fashion, both of a continuous and a discrete variable.

Work by Satoru Odake and R. Sasaki, 1

- "Infinitely many shape invariant discrete quantum mechanical systems and new exceptional orthogonal polynomials related to the Wilson and the Askey-Wilson polynomials," Phys. Lett. **B682** (2009) 130-136.
- "Another set of infinitely many exceptional (X_ℓ) Laguerre polynomials," Phys. Lett. **B684** (2010) 173-176.
- "Infinitely many shape invariant potentials and cubic identities of the Laguerre and Jacobi polynomials." J. Math. Phys. **51** (2010) 053513.
- "Exceptional Askey-Wilson type polynomials through Darboux-Crum transformations," J. Phys. **A43** (2010) 335201.
- C-L. Ho, S. Odake and R. Sasaki, "Properties of the exceptional (X_ℓ) Laguerre and Jacobi polynomials," `arXiv:0912.5477[math-ph]`.

Work by Satoru Odake and R. Sasaki, 2

- "A new family of shape invariantly deformed Darboux-Pöschl-Teller potentials with continuous ℓ," J. Phys. A **44** (2011) 195203.
- "Dual Christoffel transformations," Prog. Theor. Phys. **126** (2011) 1-34.
- "Exceptional (X_ℓ) (q)-Racah Polynomials," Prog. Theor. Phys. **125** (2011) 851-870.
- "Discrete quantum mechanics," (Topical Review) J. Phys. **A44** (2011) 353001.
- "Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials," Phys. Lett. **B702** (2011) 164-170.