Thermodynamic Bethe ansatz and analytic expansions of gluon scattering amplitudes at strong coupling

Yuji Satoh

(University of Tsukuba)

Based on

Y. Hatsuda (YITP), K. Ito (TIT), K. Sakai (YITP) and Y.S.

JHEP 1004(2010)108; 1009(2010)064; 1104(2011)100

also, to appear

1. Introduction

☆ Discovery of integrability in AdS/CFT

- spectral problem
- applications ? $\Rightarrow$ gluon scatt. amplitudes

gluon scatt. amplitudes at strong coupling

$\Uparrow$ [$\Leftarrow$ AdS/CFT]

minimal surfaces in AdS5 x S5

$\Uparrow$ [$\Leftarrow$ integrability]

thermodynamic Bethe ansatz equations

In this talk, I would like to

i) discuss MHV amplitudes of N=4 SYM at strong coupling based on underlying 2D integrable models and CFTs

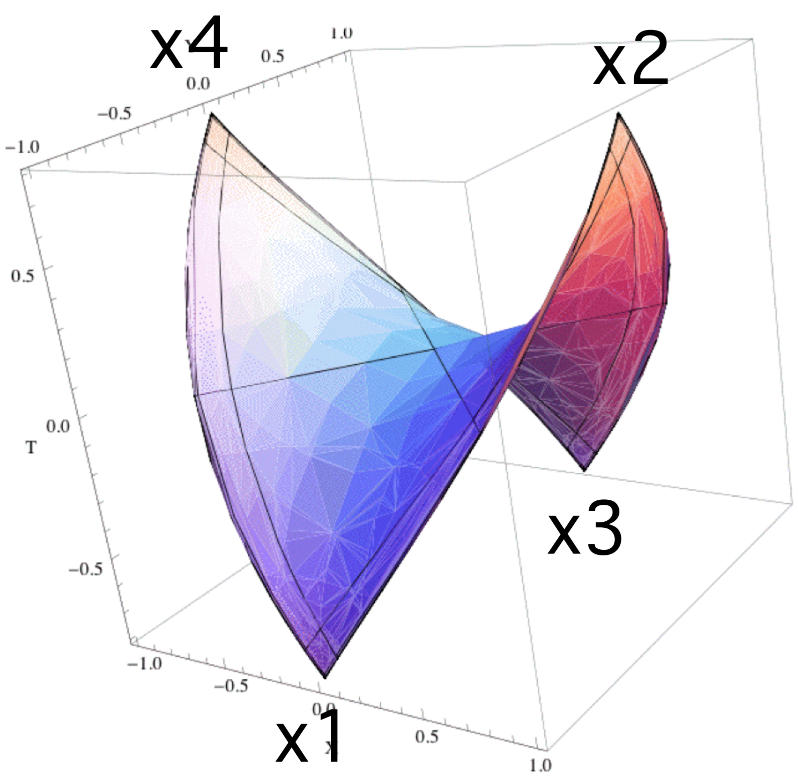
ii) in particular, derive analytic expansions around certain kinematic points

2. Gluon scattering amplitudes at strong coupling

[Alday-Maldacena '07]

amplitudes of N=4 SYM at strong coupling $\underset{\text{AdS/CFT}}{=}$ minimal surfaces in AdS

$$\mathcal{M} \sim e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})}$$



- $\mathcal{M}$: scalar part of MHV amplitudes
- λ : 't Hooft coupling
- null boundary at AdS boundary

$$x^{\mu}_{i+1} - x^{\mu}_{i} = 2\pi k^{\mu}_{i} \quad \text{(momentum of particle)}$$

$\Rightarrow$ n-pt. amplitude $\approx$ n-cusp min. surface

- 4pt. amplitude : precise agreement w/ BDS conjecture

[Bern–Dixon–Smirnov ’05]

- remainder function : deviation from BDS formula

= fn. of cross–ratios of x_a^μ $\Leftarrow$ dual conformal sym.

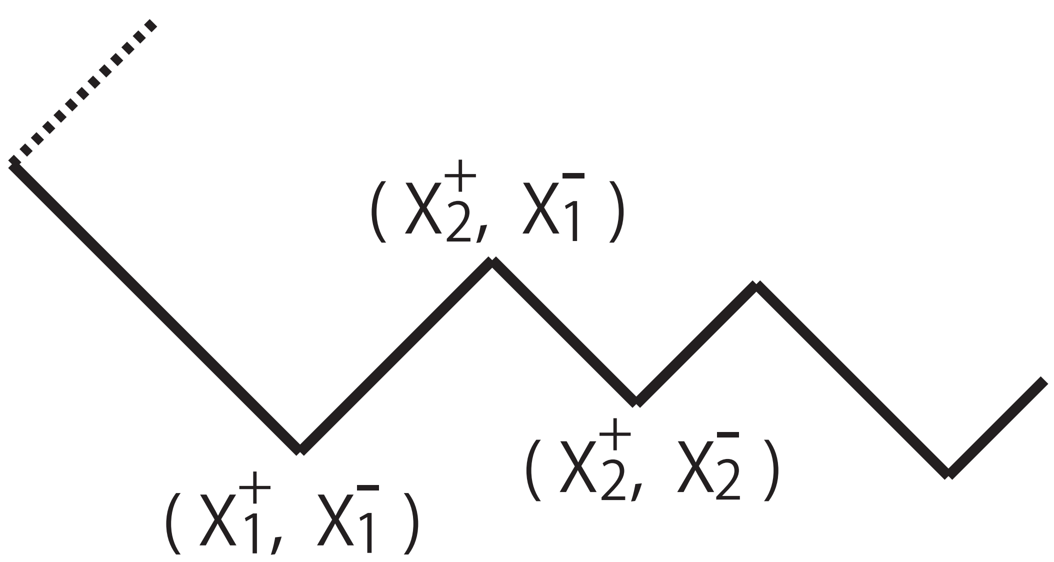
[Drummond–Henn–Smirnov–Sokatchev ’06
Drummond–Henn–Korchemsky–Sokatchev ’07]

3. Scattering amplitudes from TBA system

[Alday-Maldacena '09, Alday-Gaiotto-Maldacena '09,
Alday-Maldacena-Sever-Vieira '10, Hatsuda-Ito-Sakai-YS '10]

- difficult to construct min. surface in AdS w/ null boundary
- but, its area is obtained by using integrability w/o explicit solutions

In the following, we focus on the case of 2n-cusp min. surface in AdS3



- 4D external momenta in $R^{1,1}$
- # of cusps : even in AdS3
- 2 light-cone coord. $x^{\pm}$ at ∂(AdS)
- n : odd, for simplicity

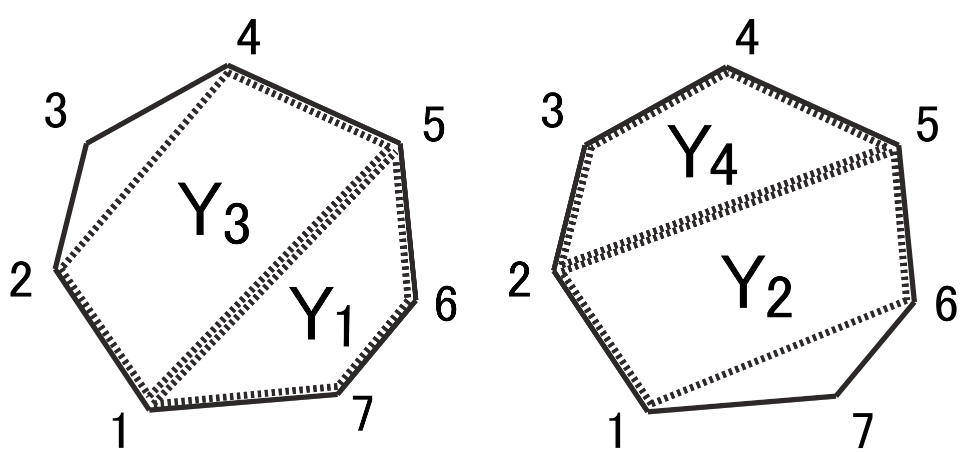
TBA for AdS3 minimal surface

- to compute amplitudes (2n-pt.), first need to solve

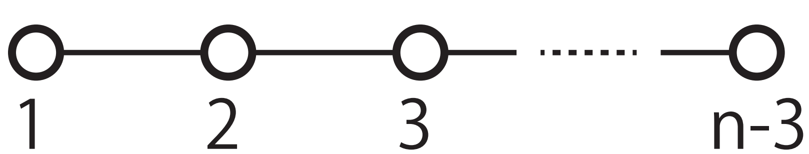
$$\log Y_s(\theta) = -m_s \cosh\theta + \sum_r K_{sr} * \log(1+Y_r)$$

(when m_s : real)

$$Y_0 = Y_{n-2} = 0 \quad (s = 1, \ldots, n-3)$$



n = 7

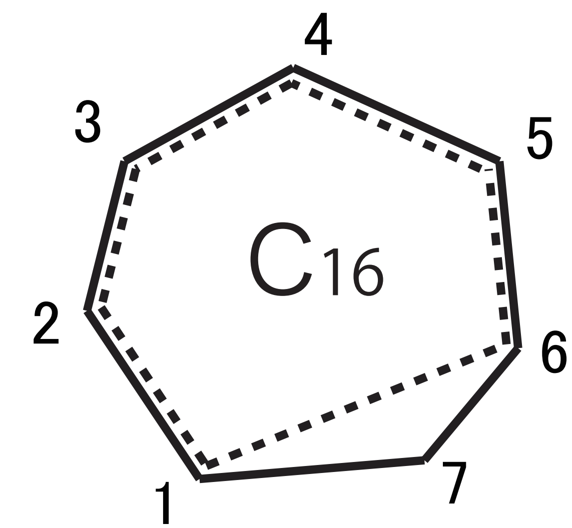


- θ : spectral parameter
- Ys : (extended) cross-ratios of $x_a^{\pm}$

  e.g.) $Y_1(-\frac{\pi i}{2}) = \frac{x_{15}^+ x_{67}^+}{x_{56}^+ x_{17}^+}, \quad Y_1(0) = \frac{x_{15}^- x_{67}^-}{x_{56}^- x_{17}^-}$
- m_s : complex (mass) param.
  ≈ shape of surface ⇔ momenta
- $K_{sr} = I_{sr}/\cosh\theta$
- I_{sr} : incidence matrix for A_{n-3}

Remainder function [overall coupling dependence : omitted]

- Once Y–fn. are obtained,



$$c_{16} = \frac{x_{23}\ x_{45}\ x_{16}}{x_{12}\ x_{34}\ x_{56}}$$

$$\begin{aligned} R &:= A - A_{BDS} \\ &= \frac{7\pi}{12}(n-2) + A_{\text{periods}} + \Delta A_{BDS} + A_{\text{free}} \end{aligned}$$

$$A_{\text{periods}} = -\frac{1}{4} m_r \, I^{-1}_{rs} \, \overline{m}_s$$

$$\Delta A_{BDS} = \frac{1}{4} \sum_{i,j=1}^{n} \log \frac{c^+_{i,j}}{c^+_{i,j+1}} \log \frac{c^-_{i-1,j}}{c^-_{i,j}} \ , \quad c^{\pm}_{i,j} = \frac{x^{\pm}_{i+2,i+1} x^{\pm}_{i+4,i+3} \cdots x^{\pm}_{j,i}}{x^{\pm}_{i+1,i} x^{\pm}_{i+3,i+2} \cdots x^{\pm}_{j,j-1}}$$

$$A_{\text{free}} = \sum_{s=1}^{n-3} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m_s \cosh\theta \log(1 + Y_s(\theta))$$

free energy of TBA system

- TBA system : solved numerically
- exact solutions $\begin{cases} m_s \to 0 & : \text{ CFT limit} \\ m_s \to \infty & \end{cases}$ regular polygon

○ solutions to TBA system : not fully investigated

○ momentum dependence : ms ↔ momentum

○ progress in analytic results at 2 loops

Any analytic results at strong coupling besides
$m_s \to 0 \ , \ \infty$??

$\Rightarrow$ we discuss expansions around CFT lim.

$$l = ML \to 0 \quad (m_s = M_s L = \tilde{M}_s M L)$$

[M: mass scale, L: inverse temp.]

4. Analytic expansion of amplitudes

[Hatsuda-Ito-Sakai-YS '11, also to appear]

- start w/observation [Hatsuda-Ito-Sakai-YS '10]

$$\boxed{\begin{array}{c}\text{TBA for AdS3}\\ \text{2n-pt. amplitude}\end{array}} \Leftarrow \boxed{\text{HSG from } \frac{\widehat{\mathrm{su}}(n-2)_2}{[\widehat{\mathrm{u}}(1)]^{n-3}}}$$

[w/ imaginary resonance param.]

- Homogeneous sine-Gordon (HSG) model for AdS3: integrable deformation of the coset CFT

[Fernandez Pousa-Gallas-Hollowood-Miramontes '96]

$$S_{HSG} = S_{\mathrm{gWZNW}} + \beta \int d^2x \ \Phi$$

Φ : comb. of weight 0 adjoint op.

$$\Delta = \bar{\Delta} = \frac{n-2}{n}, \qquad \beta = -\kappa_n M^{2(1-\Delta)}$$

A_{free} : free-energy part

- as well known,

  free energy around CFT limit $\Leftarrow$ CFT perturbation

$$A_{\text{free}} = \frac{\pi}{6} c_n + f_n^{\text{bulk}} + \sum_{k=2}^{\infty} f_n^{(k)} l^{\frac{4k}{n}}$$

$$c_n = \frac{(n-2)(n-3)}{n} \quad \text{[central charge]}$$

$$f_n^{\text{bulk}} = \frac{1}{4} \sum_{r,s=1}^{n-3} m_r I_{rs}^{-1} \bar{m}_s = -A_{\text{periods}},$$

$$f_n^{(k)} \quad \Leftarrow \quad \text{k-pt. fn. of} \quad \Phi$$

$$f_n^{(2)} = \frac{\pi}{6} C_n^{(2)} \kappa_n^2 G^2(\tilde{M}_s), \quad \langle \Phi(z)\Phi(0) \rangle = \frac{G^2(\tilde{M}_s)}{|z|^{4\Delta}}$$

$$C_n^{(2)} = 3(2\pi)^{\frac{2(n-4)}{n}} \gamma^2\left(\frac{n-2}{n}\right)\gamma\left(\frac{4-n}{n}\right), \quad \gamma(x) = \Gamma(x)/\Gamma(1-x)$$

$\Delta A_{BDS} \quad (\leftarrow \ c_{ij}^{\pm} \ )$

- can show cross-ratios $c_{ij}^{\pm}$: (surprisingly) nothing but T-fn.

$$\boxed{c_{ij}^{+} = T_{|i-j|-1}^{[i+j]}(0), \qquad c_{ij}^{-} = T_{|i-j|-1}^{[i+j+1]}(0)}$$

where $$Y_s = T_{s+1} T_{s-1}, \qquad 1 + Y_s = T_s^{[+]} T_s^{[-]}$$

$$T_s^{[k]}(\theta) := T_s\big(\theta + \frac{\pi i}{2} k\big)$$

$$\Rightarrow \qquad \Delta A_{BDS} = \frac{1}{4} \sum_{i,j=1}^{n} \log \frac{T_{|i-j|-1}^{[i+j]}}{T_{|i-j-1|-1}^{[i+j+1]}} \log \frac{T_{|i-j-1|-1}^{[i+j]}}{T_{|i-j|-1}^{[i+j+1]}}$$

T–function

- periodicity

$$\Longrightarrow \qquad T_s(\theta) = \sum_{p,q=0}^{\infty} t_s^{(p,q)} l^{(1-\Delta)(p+q)} \cosh\left(\frac{2p\theta}{n}\right) \qquad \text{[real } m_s \text{]}$$

- T–function

  ≈ g–function (boundary entropy)

[Bazhanov–Lukyanov–Zamolodchikov '94; Dorey–Runkel–Tateo–Watts '99; Dorey–Lishman–Rim–Tateo '05]

- boundary CFT perturbation

$$\Longrightarrow \frac{t_s^{(2,0)}}{t_s^{(0,0)}} = -\frac{\kappa_n G(\tilde{M}_j) B(1-2\Delta,\Delta)}{2(2\pi)^{1-2\Delta}} \left(\frac{\sin(\frac{3(s+1)\pi}{n})}{\sin(\frac{(s+1)\pi}{n})} \sqrt{\frac{\sin(\frac{\pi}{n})}{\sin(\frac{3\pi}{n})}} - \sqrt{\frac{\sin(\frac{3\pi}{n})}{\sin(\frac{\pi}{n})}} \right)$$

$$t_s^{(0,0)} = \frac{\sin\left(\frac{(s+1)\pi}{n}\right)}{\sin(\frac{\pi}{n})}$$

Combining all, remainder fn. is expanded as

$$R_{2n} = R_{2n}^{(0)} + l^{\frac{8}{n}} R_{2n}^{(4)} + \mathcal{O}(l^{\frac{12}{n}})$$

$$R_{2n}^{(0)} = \frac{\pi}{4n}(n-2)(3n-2) - \frac{n}{2}\sum_{s=1}^{\frac{n-3}{2}} \log^2\left(\frac{\sin(\frac{(s+1)\pi}{n})}{\sin(\frac{s\pi}{n})}\right)$$

$$R_{2n}^{(4)} = \frac{\pi}{6} C_n^{(2)} \kappa_n^2 G^2(\tilde{M}_j) - \frac{n}{4}\left[\sum_{s=1}^{\frac{n-3}{2}} A_{n,s} - 2\left(\frac{t_{(n-3)/2}^{(2,0)}}{t_{(n-3)/2}^{(0,0)}}\right)^2 \sin^2\left(\frac{\pi}{n}\right)\right]$$

$$A_{n,s} := \left[\left(\frac{t_{s-1}^{(2,0)}}{t_{s-1}^{(0,0)}}\right)^2 + \left(\frac{t_s^{(2,0)}}{t_s^{(0,0)}}\right)^2\right]\cos\left(\frac{2\pi}{n}\right) - \frac{2t_{s-1}^{(2,0)} t_s^{(2,0)}}{t_{s-1}^{(0,0)} t_s^{(0,0)}}$$
$$+ \left[\left(\frac{t_{s-1}^{(2,0)}}{t_{s-1}^{(0,0)}}\right)^2 - \left(\frac{t_s^{(2,0)}}{t_s^{(0,0)}}\right)^2 - 4\left(\frac{t_{s-1}^{(0,4)}}{t_{s-1}^{(0,0)}} - \frac{t_s^{(0,4)}}{t_s^{(0,0)}}\right)\right] \log\left(\frac{t_s^{(0,0)}}{t_{s-1}^{(0,0)}}\right)$$

- $t_s^{(2,0)},\ t_s^{(0,4)} \Leftarrow t_1^{(2,0)} \propto \kappa_n G(\tilde{M}_s)$

  T-system

- still, need to find $\kappa_n G \Leftrightarrow m_s \Leftrightarrow$ (momenta)

  [mass-coupling relation]

- this can be done for single mass cases

  [Zamolodchikov '95; Fateev '94]

  - $M_s = \delta_{s,r} M$: RSOS, SU(2) coset
  - $M_1 = M_{n-3} = M,$ (others) = 0 : magnonic $T_{(n-3)/2}$

- For example, for 10-pt.,

  these are enough to completely fix $\kappa_n G(\tilde{M}_s)$

10 pt. remainder function

$$R_{10} = R_{10}^{(0)} + R_{10}^{(4)} \cdot l^{8/5} + \mathcal{O}(l^{12/5})$$

$$[\; l = ML,\;\; m_s = \tilde{M}_s l \;]$$

$$R_{10}^{(0)} = \frac{39}{20}\pi - \frac{5}{2}\log^2\bigl(2\cos\frac{\pi}{5}\bigr)$$

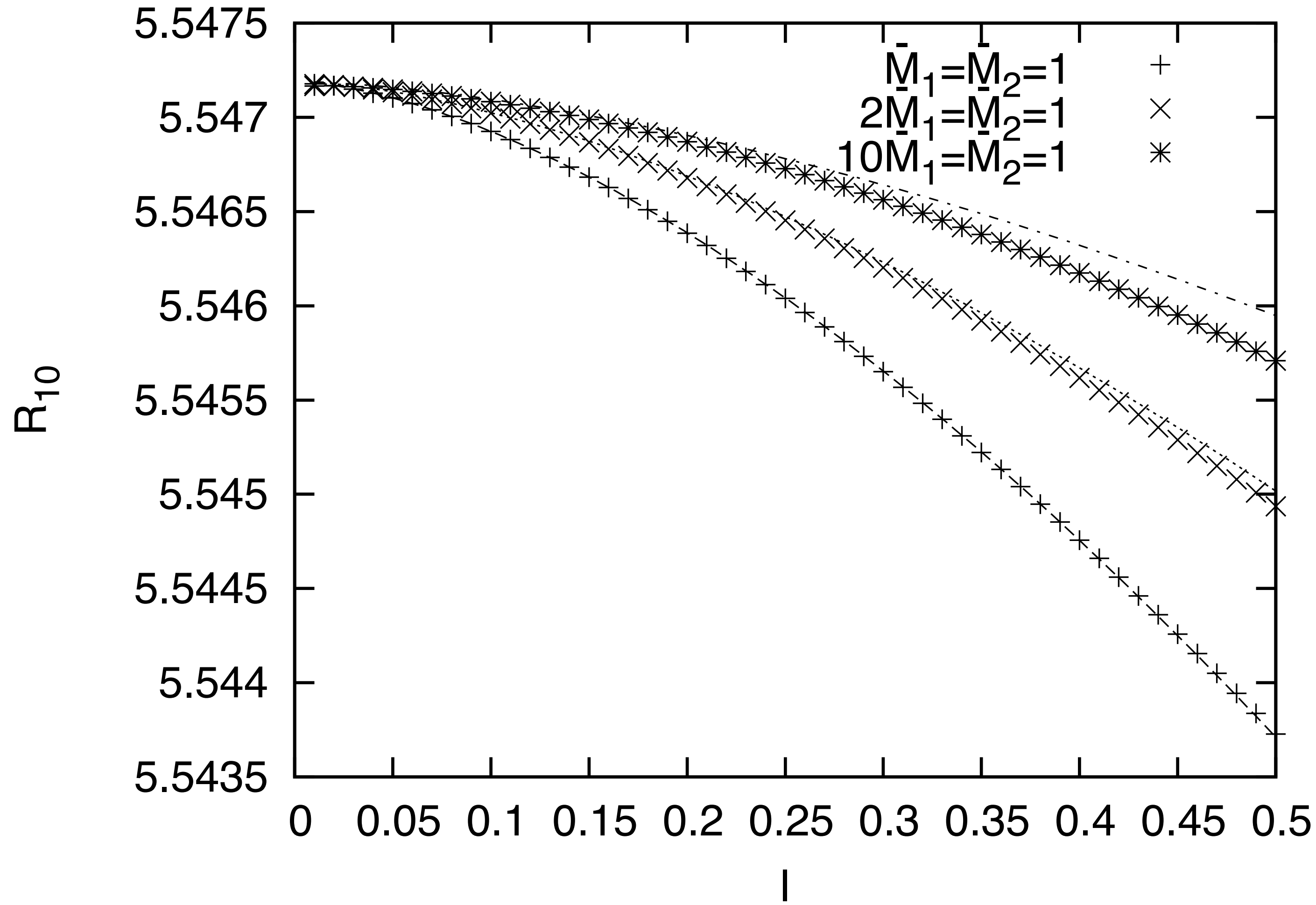
$$R_{10}^{(4)} = \bigl(-\frac{1}{5}\tan\frac{\pi}{5} + C_1\bigr)\cdot\bigl|t_1^{(2,0)}\bigr|^2$$

$$t_1^{(2,0)} = C_2(\tilde{M}_1^{4/5} + \tilde{M}_2^{4/5} - C_3\tilde{M}_1^{2/5}\tilde{M}_2^{2/5})$$

$$C_1 = 20\cos^4\bigl(\frac{2\pi}{5}\bigr)\Bigl(1 - 5^{-1/2}\log\bigl(2\cos\frac{\pi}{5}\bigr)\Bigr)$$

$$C_2 = \frac{1}{4\cdot 6^{1/5}}\Gamma\bigl(-1/5\bigr)\Bigl[10\cos\frac{\pi}{5}\gamma\bigl(3/5\bigr)\gamma\bigl(4/5\bigr)\Bigr]^{1/2}$$

$$C_3 = 2 - \bigl(\frac{3}{\pi^2}\bigr)^{1/5}\gamma\bigl(1/4\bigr)^{4/5}$$



l- dependence of 10–pt. remainder function $[\, m_s = \tilde{M}_s l \, e^{i\varphi_s} \,]$

- dashed lines : $R_{10}^{(0)} + R_{10}^{(4)} l^{8/5}$
- $\varphi_1 = \pi/20$, $\varphi_2 = \pi/5$
- good agreement w/ numerics

7. Comparison with 2-loop results

2-loop results

[Heslop-Khoze' 10 ; Gaiotto-Maldacena-Sever-Vieira ' 11]

- analytic results at 2 loops are known for AdS3 amplitudes
- for the same cross-ratios, has similar expansion

$$R_{2n}^{\text{2-loop}} = R_{2n}^{\text{2-loop}\,(0)} + l^{\frac{8}{n}} R_{2n}^{\text{2-loop}\,(4)} + \mathcal{O}(l^{\frac{12}{n}})$$

rescaled remainder fn.

[Brandhuber-Heslop-Khoze-Travaglini ' 09]

- for comparison, useful to define

$$\overline{R}_{2n} := \frac{R_{2n} - R_{2n,\text{UV}}}{R_{2n,\text{UV}} - (n-2)R_6}$$

  - normalized s.t. $\overline{R}_{2n} \to 0 \quad (l \to 0)$
    $\to -1 \quad (l \to \infty)$

- dependence on m_s (mometum) :

  encoded in $t_1^{(2,0)} \propto \kappa_n G(\tilde{M}_s)$ at leading order

$\Rightarrow$ ratio of $\overline{R}_{2n}$: number

$$\frac{\bar{R}_8^{\text{strong}}}{\bar{R}_8^{\text{2-loop}}} \approx 1.0257\,, \qquad \frac{\bar{R}_{10}^{\text{strong}}}{\bar{R}_{10}^{\text{2-loop}}} \approx 0.9841$$

$$\frac{\bar{R}_{12}^{\text{strong}}}{\bar{R}_{12}^{\text{2-loop}}} \approx 0.9609\,, \qquad \frac{\bar{R}_{14}^{\text{strong}}}{\bar{R}_{14}^{\text{2-loop}}} \approx 0.9463$$

$$\frac{\bar{R}_{16}^{\text{strong}}}{\bar{R}_{16}^{\text{2-loop}}} \approx 0.9366\,, \qquad \frac{\bar{R}_{18}^{\text{strong}}}{\bar{R}_{18}^{\text{2-loop}}} \approx 0.9297$$

$$\vdots$$

$$\frac{\bar{R}_{n\gg 1}^{\text{strong}}}{\bar{R}_{n\gg 1}^{\text{2-loop}}} \approx 0.905 - \frac{0.118}{n}$$

surprisingly close

10. Summary

◇ Gluon scatt. amplitudes of N=4 SYM at strong coupling

$\Uparrow$ minimal surfaces in AdS [$\Leftarrow$ AdS/CFT]

$\Uparrow$ TBA (type) eq., HSG model [$\Leftarrow$ integrability]

◇ Underlying 2D integrable (HSG) model

$\Rightarrow$ analytic expansions of amplitudes around CFT lim. [regular polygon]

- $A_{\text{free}} \Leftarrow$ bulk CFT perturbation
- $\Delta A_{BDS} \Leftarrow$ T–function
- T–/Y–fn. $\Leftarrow$ g–fn. $\Leftarrow$ boundary CFT perturbation

◇ Rescaled remainder fn. at strong coupling, 2 loops: close

[cf. ODE/IM, prof. Bazhanov's talk]

- Why minimal surface ⇔ TBA ?

  T-functions ⇔ g-functions (boundary entropy) ?

- Why strong coupling ⇔ 2 loops : so close ?

- General case : AdS4 ⇐ HSG model for $\dfrac{\widehat{\mathrm{su}}(m-4)_4}{[\widehat{\mathrm{u}}(1)]^{m-5}}$ [m-pt.]

  AdS5 ⇐ ??

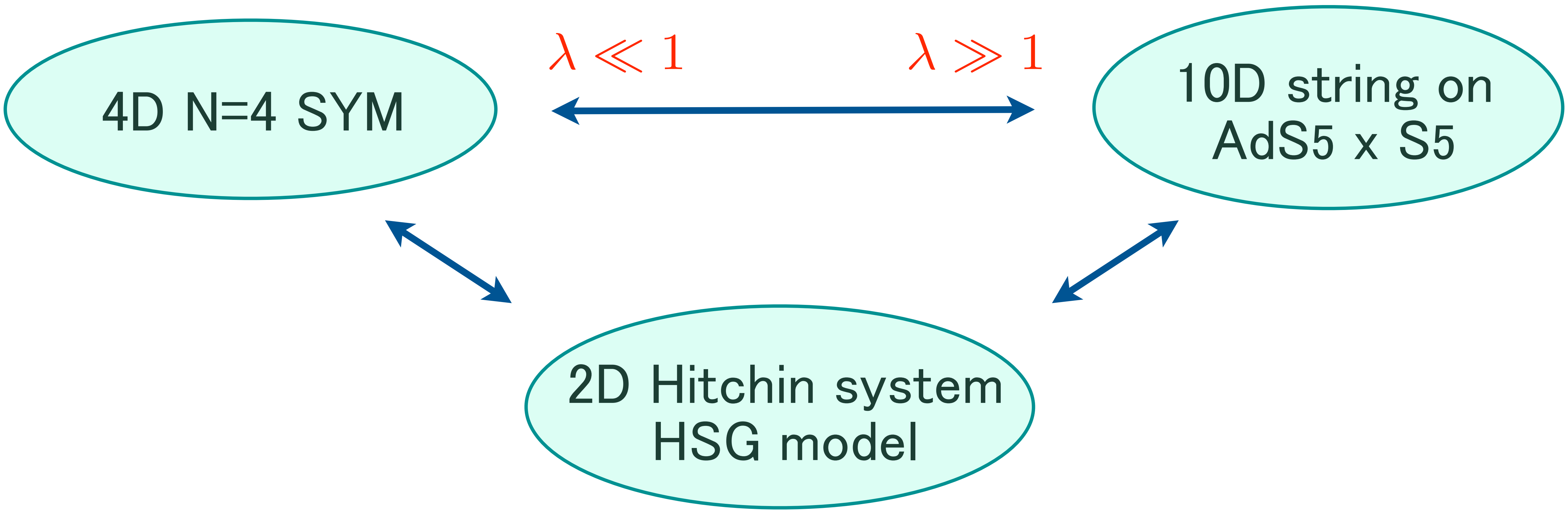
  - multi-parameter integrable deformation of CFT
  - form of $\kappa_n \Phi(\tilde{M}_s)$, mass-coupling relation

- Higher order terms ?

- Strong coupling corrections ?

[cf. Alday-Gaiotto-Maldacena-Sever-Vieira '10]

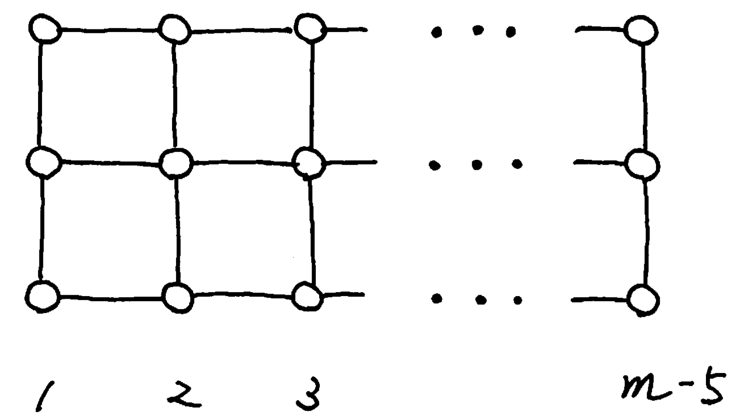
⋮



[λ : 't Hooft coupling]

Y-system for min. surface in AdS5

- for m-cusp min. surface in AdS5



$$\frac{Y_{2,s}^{-}Y_{2,s}^{+}}{Y_{1,s}Y_{3,s}} = \frac{(1+Y_{2,s+1})(1+Y_{2,s-1})}{(1+Y_{1,s})(1+Y_{3,s})}$$

$$\frac{Y_{3,s}^{-}Y_{1,s}^{+}}{Y_{2,s}} = \frac{(1+Y_{3,s+1})(1+Y_{1,s-1})}{1+Y_{2,s}}$$

$$\frac{Y_{1,s}^{-}Y_{3,s}^{+}}{Y_{2,s}} = \frac{(1+Y_{1,s+1})(1+Y_{3,s-1})}{1+Y_{2,s}}$$

$$Y_{a,s}^{\pm}(\theta) = Y_{a,s}(\theta \pm \frac{\pi}{4} i), \quad s = 1, ..., m-5$$