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Conformal blocks and the Calogero-Sutherland model

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Calogero-Sutherland model and 2d CFT's

- Laughlin wavefunction, $c=1$ CFT and CS model [Haldane 90's, ... Haldane, Bernevig 07]
- Matrix models and collective field representation for CS model [Jevicki, 92,...]
- Spinons in $su(2)_{k=1}$ WZW model [Bernard, Pasquier, DS, 94]
- CS and singular vectors of Virasoro algebra [Awata, Matsuo, Odake, Shiraishi, 95; Arnaudon, Avan, Frappat, Ragoucy, Shiraishi, 06]
- Quantum hydrodynamics, CS and Benjamin-Ono [Abanov, Wiegmann, 05; this morning's talk]
- SLE, CFT and CS model [Cardy, 04; Cardy, Doyon, 07; Dubedat, 06]
- FQHE with pairing properties, CFT's and CS model [Nayak, Wilczek, 96; Haldane, Bernevig 07; Estienne, Bernevig, Santachiara, 10]
- AGT conjecture and CS model [Alba, Fateev, Litvinov, Tarnopolsky 10; V. Fateev's talk; Belavin, Belavin 11, ...]

...

Conformal blocks of some 2d CFT's

$$\begin{aligned} & \langle \Phi_{12}(z_1) \cdots \Phi_{12}(z_N) \Phi_{21}(w_1) \cdots \Phi_{21}(w_M) \rangle \\ &= \sum_{\lambda} \langle 0 | \Phi_{12}(z_1) \cdots \Phi_{12}(z_N) | \lambda \rangle \langle \lambda | \Phi_{21}(w_1) \cdots \Phi_{21}(w_M) | 0 \rangle \end{aligned}$$

FQHE states with non-abelian statistics

$$\begin{aligned} g &\rightarrow -g \\ b &\rightarrow ib \end{aligned}$$

[Estienne, Bernevig, Santachiara, 10]

AGT conjecture
(Nekrasov's partition function $\sim$
Liouville conformal blocks)

[Alba, Fateev, Litvinov, Tarnopolsky 10]

V. Fateev's talk

Integrable structure of the CS model

Summary

- CS Hamiltonian and the degenerate fields in CFT
- duality of the conformal blocks
- Ising CFT and FQHE states
- AFLT Hamiltonians for generic Virasoro models
- WA_{k-1} models

Calogero-Sutherland Hamiltonian

Trigonometric CS model: set of N commuting Hamiltonians for N particles on a circle:

$$H_1^g = \mathcal{P} = \sum_{i=1}^N z_i \partial_i$$

$$H_2^g = H^g = \sum_{i=1}^N (z_i \partial_i)^2 - g(g-1) \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2}$$

$$z_j = e^{2i\pi x_j / L}$$

$$H_3^g = \sum_{i=1}^N (z_i \partial_i)^3 + \frac{3}{2}g(1-g) \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2} (z_i \partial_i - z_j \partial_j).$$

...

Two different **boundary conditions** for the wave functions:

$$\Psi^+(z) = \Delta^g(z) F^+(z) \quad \text{or} \quad \Psi^-(z) = \Delta^{1-g}(z) F^-(z)$$

$$\Delta^\gamma(z) = \prod_{i < j} (z_i - z_j)^\gamma$$

Polynomial eigenfunctions (Jack symmetric polynomials) $\longleftrightarrow$ abelian statistics

$$\Psi_\lambda^+(z) = \Delta^g(z) J_\lambda^{1/g}(z)$$

$$\Psi_\lambda^-(z) = \Delta^{1-g}(z) J_\lambda^{1/(1-g)}(z)$$

Jack polynomials: eigenfunctions of the Hamiltonian

$$\mathcal{H}^\alpha = \sum_{i=1}^N (z_i \partial_i)^2 + \frac{1}{\alpha} \sum_{i < j}^N \frac{z_i + z_j}{z_{ij}} (z_i \partial_i - z_j \partial_j)$$

$$\alpha^{-1} = g \quad \text{or} \quad 1 - g$$

$$\mathcal{E}_\lambda^\alpha = \sum_i^N \lambda_i \left[\lambda_i + \frac{1}{\alpha} (N + 1 - 2i) \right]$$

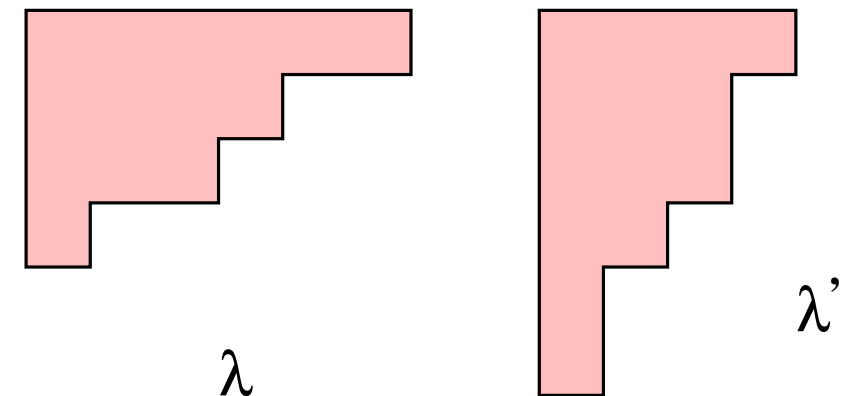
characterized by **partitions** λ with λ_i integers: $\lambda_1 \geq \dots \geq \lambda_N \geq 0$

Duality $g \rightarrow 1/g$: [Stanley 89; Macdonald 88; Gaudin 92]

$$[\mathcal{H}^{1/g} + g \mathcal{H}^g + C(N, M)] \prod_{i=1}^N \prod_{j=1}^M (1 + z_i w_j) = 0$$

$$\prod_{i,j} (1 + z_i w_j) = \sum_{\lambda} J_{\lambda}^{1/g}(z) J_{\lambda'}^g(w)$$

Dual partitions:



Degenerate fields in CFT

Virasoro models with central charge : $c = 1 - 6 \frac{(g-1)^2}{g}$

Degenerate field with dimensions : $\Delta_{(r|s)} = \frac{1}{4} \left(\frac{r^2-1}{g} + (s^2-1)g + 2(1-rs) \right)$

Two second-level degenerate fields : $(L_{-1}^2 - gL_{-2}) \Phi_{(1|2)} = 0$, $\left(L_{-1}^2 - \frac{1}{g}L_{-2} \right) \Phi_{(2|1)} = 0$

When inserted in correlation function, the null-vector conditions translate into **differential equations**:

$$\mathcal{O}^g(z) \langle \Phi_{(1|2)}(z) \Phi_{\Delta_1}(z_1) \dots \Phi_{\Delta_N}(z_N) \rangle = 0$$

with

$$\mathcal{O}^g(z) = \frac{\partial^2}{\partial z^2} - g \left(\sum_{j=1}^N \frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j} \frac{\partial}{\partial z_j} \right)$$

Conformal blocks and the duality

Consider the **dressed conformal blocks** :

$$\mathcal{F}_{M,N}^{a,b}(w; z) \equiv \langle \Phi_{21}(w_1) \cdots \Phi_{21}(w_M) \Phi_{12}(z_1) \cdots \Phi_{12}(z_N) \rangle_{a,b} \prod_{1 \leq i < j}^M w_{ij}^{2\tilde{h}} \prod_{i,j} (w_i - z_j)^{1/2} \prod_{1 \leq i < j}^N z_{ij}^{2h}$$

$$h = \Delta_{12} = \frac{3g}{4} - \frac{1}{2}, \quad \tilde{h} = \Delta_{21} = \frac{3}{4g} - \frac{1}{2}$$

CS action on the conformal blocks (M=0 case: **[Cardy 04]**)

Duality :

$$\left[h^\alpha(z) + g h^{\tilde{\alpha}}(w) \right] \mathcal{F}_{M,N}^{a,b}(w; z) = 0$$

with dual CS Hamiltonians

$$h^\alpha(z) \equiv \mathcal{H}^\alpha(z) - \mathcal{E}_0^\alpha + \left(\frac{N-2}{\alpha} - 1 \right) [\mathcal{P}(z) - \mathcal{P}_0] - \frac{NM(M-2)}{4},$$

$$h^{\tilde{\alpha}}(w) \equiv \mathcal{H}^{\tilde{\alpha}}(w) - \mathcal{E}_0^{\tilde{\alpha}} + \left(\frac{M-2}{\tilde{\alpha}} - 1 \right) [\mathcal{P}(w) - \mathcal{P}'_0] - \frac{NM(N-2)}{4},$$

and dual coupling constants

$$\alpha^{-1} = 1 - g, \quad \tilde{\alpha}^{-1} = 1 - g^{-1}$$

non-abelian generalization of the
Stanley-Macdonald-Gaudin duality

$$\mathcal{F}_{M,N}^{a,b}(w; z) = \sum_{\lambda} P_{\lambda'}^{\tilde{\alpha},a}(w) P_{\lambda}^{\alpha,b}(z)$$

[Estienne, Bernevig, Santachiara 10]
for Z_k parafermionic CFTs

Ising CFT and the Moore-Read FQHE wave-function

[Moore, Read, 91]

$$\Psi = \Phi_{12} \quad \text{electron}$$

$$\sigma = \Phi_{21} \quad \text{quasihole}$$

$$g = 4/3.$$

$$\alpha = 1/(1 - g) = -3$$

The electron eigenfunction is **monovalued**:

$$\text{Moore-Read wavefunction} \sim \prod_{i < j} z_{ij}^2 \langle \Psi(z_1) \dots \Psi(z_N) \rangle = \prod_{i < j} z_{ij}^2 \text{Pf} \left(\frac{1}{z_{ij}} \right) = \prod_{i < j} z_{ij} J_{\lambda_0}^{-3}(z)$$

$$\lambda_0 = [N - 2, N - 2, N - 4, N - 4, \dots, 0, 0]$$

with **clustering properties** (it vanishes when a cluster of 3 particles come together):

eigenfunction of a three-body Hamiltonian

$$H = \sum_{i \neq j \neq k} \delta^{(2)}(x_i - x_j) \delta^{(2)}(x_j - x_k)$$

Generic (k, r) clustering properties of Jack polynomials for coupling constant $\alpha = -(k + 1)/(r - 1)$

[Feigin, Miwa, Jimbo, Mukhin, 02]

- the coefficients of the Jack polynomials diverge except for **admissible partitions**

$$(k, r, N)\text{-admissible partition } \lambda : \quad \lambda_i - \lambda_{i+k} \geq r \quad (1 \leq i \leq N - k)$$

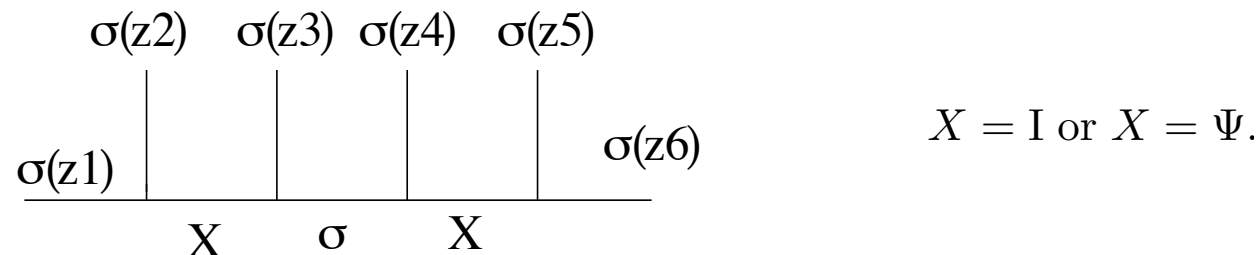
$$J_{\lambda}^{-(k+1)/(r-1)}(z_1, \dots, z_n) \text{ vanishes when } z_1 = z_2 = \dots = z_{k+1}$$

Ising CFT and the Moore-Read FQHE wave-function

quasihole wave-function: $\sim \Psi(z)_a \equiv \prod_{i < j} z_{ij}^{3/8} \langle \sigma(z_1) \dots \sigma(z_M) \rangle_a$

[Nayak, Wilczek, 96]

it is **multivalued**, with the $2^{M/2-1}$ conformal blocks corresponding to the different fusion channels



→ wave-function with non-abelian braiding properties

$$\Psi(z)_a = \prod_{i < j} z_{ij}^{1/4} F(z)_a = \prod_{i < j} z_{ij}^{1-g} F(z)_a$$

with $F(z)_a \sim c_{a1} + c_{a2} \sqrt{z_{ij}}$ for $z_i \rightarrow z_j$.

- **non-polynomial eigenfunctions** of the CS Hamiltonian with non-abelian monodromy
- symmetric generalization of **hypergeometric functions** ~ [Kaneko, 93; Forrester, 92]
- can be represented as Coulomb integrals [Dotsenko, Fateev, 84]

Duality and the non-polynomial wave-functions

How to characterize an arbitrary excited “partition” λ (λ_i generically not integers) ?

$$\mathcal{F}_{M,N}^{a,b}(w; z) = \sum_{\lambda} P_{\lambda'}^{\tilde{\alpha},a}(w) P_{\lambda}^{\alpha,b}(z)$$

- **ground state (M=0):** $\langle \Phi_{12}(z_1) \cdots \Phi_{12}(z_N) \rangle_a \prod_{i < j} z_{ij}^{2h} \longrightarrow$ smallest “partition”

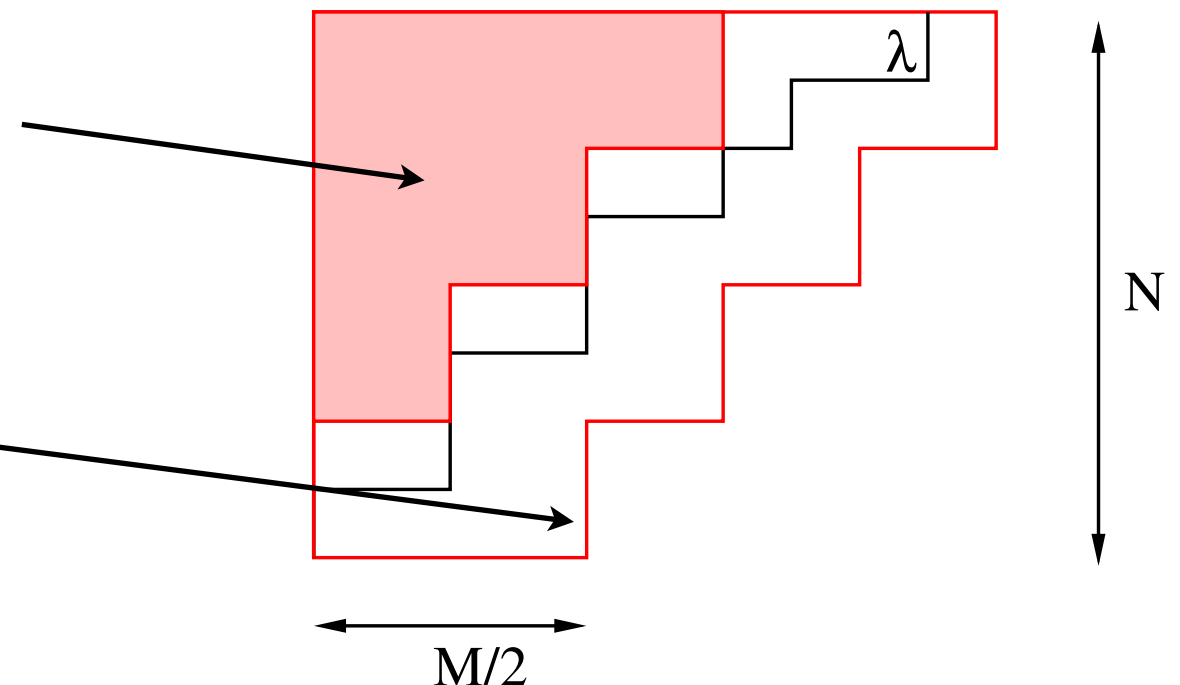
special conformal block with X=I: $z_1^{2h(N-2)} z_2^{2h(N-2)} z_3^{2h(N-4)} z_4^{2h(N-4)} \cdots z_{N-1}^0 z_N^0 + \cdots$

smallest “partition”

$$\lambda_{2i-1}^0 = \lambda_{2i}^0 = 2h(N - 2i)$$

largest “partition”

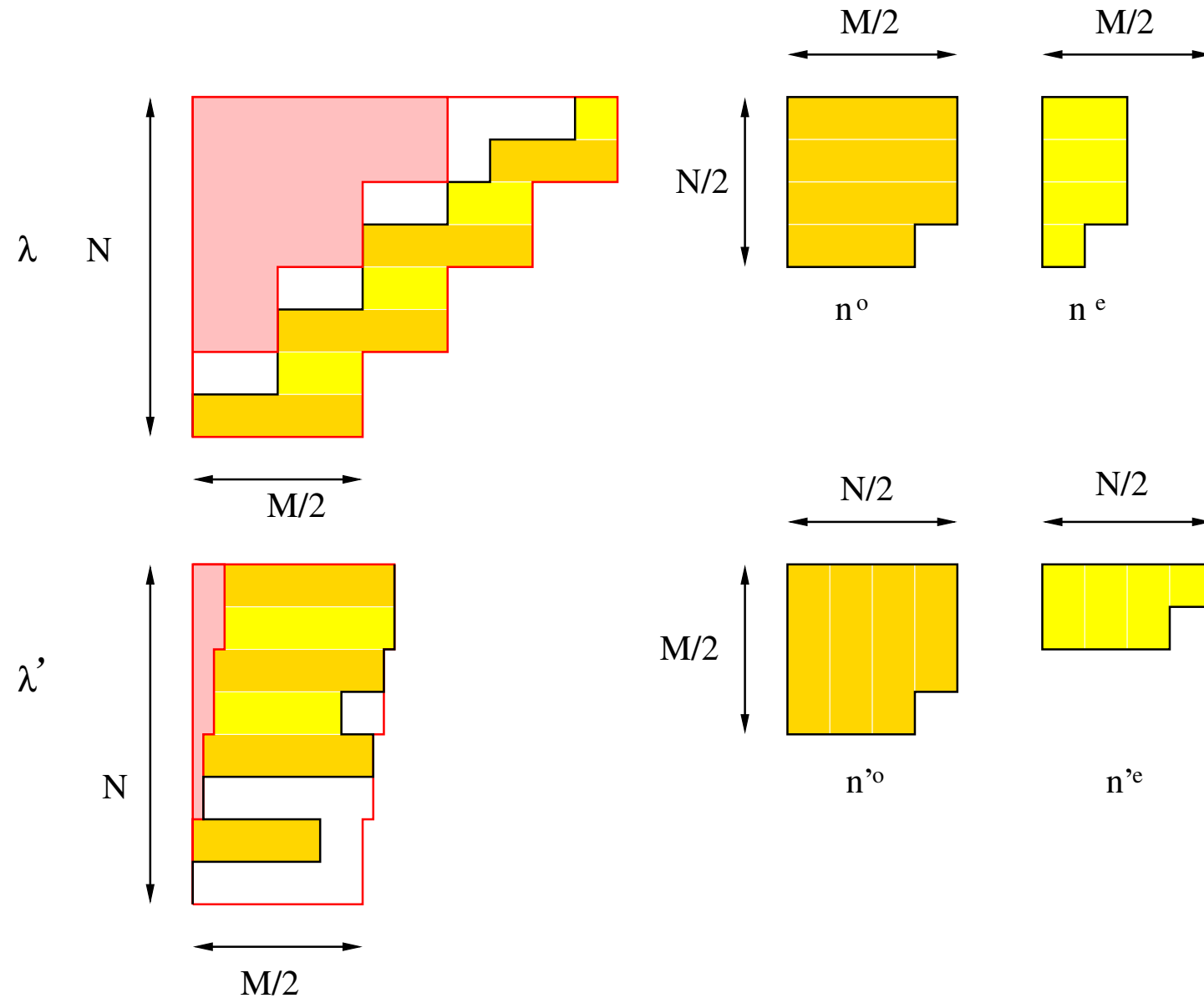
$$\Lambda_i^0 = \lambda_i^0 + \frac{M}{2}$$



- Ising: $2h=1 \longrightarrow$ λ is a partition satisfying the (2,2) admissibility condition $\lambda_i - \lambda_{i+2} \geq 2$

Duality and the non-polynomial wave-functions

An excited state λ is characterized by **two partitions** n_e and n_o (reminiscent of AGT conjecture):



for the partition n' **dual** to n :

$$b(n) \equiv 2 \sum_i (i-1)n_i = \sum_i n'_j (n'_j - 1)$$

$$|n| \equiv \sum_i n_i = |n'|$$

$$\mathcal{E}_\lambda^\alpha = \sum_{i=1}^N \lambda_i [\lambda_i + (1-g)(N+1-2i)]$$

$$\mathcal{E}_{\lambda'}^{\tilde{\alpha}} = \sum_{j=1}^M \lambda'_j [\lambda'_j + (1-g^{-1})(M+1-2j)]$$

$$\mathcal{E}_\lambda^\alpha = [b(n'^o) + b(n'^e)] - g [b(n^o) + b(n^e)] + ((1-g)N - M + g)(|n^o| + |n^e|) + 2(g-1)|n^e| + \mathcal{E}_{(M/2)^N}^\alpha$$

$$\mathcal{E}_{\lambda'}^{\tilde{\alpha}} = [b(n^o) + b(n^e)] - \frac{1}{g} [b(n'^o) + b(n'^e)] + \frac{(2-g)M + 2g - 3}{g} (|n'^e| + |n'^o|) + \frac{2(g-1)}{g} |n'^o| + \mathcal{E}_0^{\tilde{\alpha}}$$

additive spectrum??

u(1) x Virasoro models

introduce a u(1) component $\longrightarrow$ electromagnetic current for FQHE $J(z) = i\partial\phi(z)$

Heisenberg algebra : $[a_n, a_m] = n\delta_{n+m,0}$

Virasoro algebra : $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$

Feigin-Fuchs representation : $L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} : b_{n-m} b_m : - \alpha_0(n+1)b_n$ $2\alpha_0 = \sqrt{\frac{2}{g}} - \sqrt{2g}$ $c = 1 - 12\alpha_0^2$

Degenerate fields dressed by u(1) vertex operators :

$$V(z) \equiv \Phi_{12}(z) e^{i\sqrt{\frac{g}{2}}\phi(z)}, \quad \tilde{V}(w) \equiv \Phi_{21}(w) e^{i\frac{1}{\sqrt{2g}}\phi(w)}$$

Consider generic correlation functions :

$$\begin{aligned} f_\mu^+(z_1, z_2, \dots, z_N) &= \langle \mu | V(z_1) V(z_2) \cdots V(z_N) | P \rangle & |\mu\rangle & : \text{generic state (primary or descendant)} \\ f_\mu^-(z_1, z_2, \dots, z_N) &= \langle P | V(z_1) V(z_2) \cdots V(z_N) | \mu \rangle & |P\rangle & : \text{primary state} \end{aligned}$$

Translate the CS action on states :

$$H_n^g f_\mu^\pm(z_1, z_2, \dots, z_N) = \sum_\nu [I_{n+1}^\pm(g)]_{\mu,\nu} f_\nu^\pm(z_1, z_2, \dots, z_N)$$

CS integrals of motions and the Hilbert space of $u(1) \times \text{Virasoro}$

- **second-order** null vector condition: $(L_{-1}^2 - gL_{-2}) V = 0$

→ second order CS hamiltonian:

$$I_3^{(\pm)}(g) = 2(1-g) \sum_{m \geq 1} m a_{-m} a_m \pm \sqrt{2g} \sum_{m \neq 0} a_{-m} L_m \pm \sqrt{\frac{g}{2}} \left(\sum_{m,k \geq 1} a_{-m-k} a_m a_k + a_{-m} a_{-k} a_{m+k} \right)$$

- null vector condition at **level 3**: $(L_{-1} + 3\sqrt{g/2}a_{-1})(L_{-1}^2 - gL_{-2}) V = 0$

→ third order CS hamiltonian:

$$I_n^{(\pm)}(g) \propto I_n^{(\mp)}(1/g)$$

$$\begin{aligned} I_4^{\pm}(g) = & -g \sum_{m>0} L_{-m} L_m \\ & - \frac{3}{2}g \sum_{m,p>0} (2L_{-p}a_{-m}a_{p+m} + 2a_{-m-p}a_m L_p + a_{-m}a_{-p}L_{p+m} + L_{-m-p}a_m a_p) \\ & \pm \frac{3}{2}\sqrt{2g}(g-1) \sum_{m>0} m(a_{-m}L_m + L_{-m}a_m) \pm 3\sqrt{2g}(g-1) \sum_{m,p>0} m(a_{-m}a_{-p}a_{m+p} + a_{-m-p}a_m a_p) \\ & - \frac{1}{2}gL_0^2 - 3gL_0 \sum_{m>0} a_{-m}a_m + \sum_{m \geq 1} \left[\frac{1}{2}(9g-5-5g^2)m^2 - \frac{1}{2}(g-1)^2 \right] a_{-m}a_m \\ & - \frac{g}{8} \sum_{\substack{m_1+m_2+m_3+m_4=0 \\ m_i \neq 0}} : a_{m_1} a_{m_2} a_{m_3} a_{m_4} : \end{aligned}$$

[Alba, Fateev, Litvinov, Tarnopolsky,10]
for Liouville $g \rightarrow -g$

Jack polynomials and the Hilbert space of $u(1) \times$ Virasoro

- rotate the boson basis: $c_m = \frac{1}{\sqrt{2}} (a_m + b_m)$, $\tilde{c}_m = \frac{1}{\sqrt{2}} (a_m - b_m)$ [AFLT, 10; Belavin and Belavin, 11]

- introduce the one-component bosonised CS Hamiltonians:

$$\mathcal{I}_3^\pm(c; g) = (1 - g) \sum_{m>0} m c_{-m} c_m \pm \sqrt{g} \sum_{m,k>0} (c_{-m-k} c_m c_k + c_{-m} c_{-k} c_{m+k})$$

[Jevicki, 91]
[Awata, Matsuo, Odake, Shiraishi, 95]

$$\begin{aligned} \mathcal{I}_4^\pm(c; g) = & \pm \left(\frac{3g}{2} - g^2 - 1 \right) \sum_{m>0} m^2 c_{-m} c_m - \frac{g}{4} \sum_{\substack{m_1+m_2+m_3+m_4=0 \\ m_i \neq 0}} : c_{m_1} c_{m_2} c_{m_3} c_{m_4} : \pm \\ & \pm 3\sqrt{g}(g-1) \sum_{m,l>0} m (c_{-m-l} c_m c_l + c_{-m} c_{-l} c_{m+l}) . \end{aligned}$$

- classical limit $g \rightarrow 0$, $v = \sqrt{g} \partial \phi$ and $\mathcal{I}_n \rightarrow g \mathcal{I}_n \longrightarrow$ Benjamin-Ono hierarchy:

$$\mathcal{I}_2 = \int dx \frac{1}{2} v^2 ,$$

$$\mathcal{I}_3 = \int dx \left(\frac{1}{3} v^3 + \frac{1}{2} v H(v_x) \right) ,$$

$$\mathcal{I}_4 = \int dx \left(\frac{1}{4} v^4 + \frac{1}{4} v_x^2 + \frac{3}{4} v^2 H(v_x) \right)$$

$H(f)$ is the Hilbert transform of the function f .

$$v_x = \partial_x v$$

CS $\longleftrightarrow$ BO: [Abanov, Wiegmann, 05]

Jack polynomials and the Hilbert space of $u(1) \times \text{Virasoro}$

At $g=1$ the CS Hamiltonian is a sum of two copies of one-component CS models [Belavin and Belavin, 11]
(up to zero modes):

$$I_3^+(1) = \mathcal{I}_3(c) + \mathcal{I}_3(\tilde{c}) + \sqrt{2}b_0(L_0(c) - L_0(\tilde{c}) - a_0b_0)$$

$$I_4^+(1) = \mathcal{I}_4(c) + \mathcal{I}_4(\tilde{c}) - b_0D(c, \tilde{c}, b_0)$$

The states can be constructed with the help of Schur polynomials:

$$|n^o, n'^e; q\rangle = S_{n^o}(c)S_{n^e}(\tilde{c})|q\rangle + S_{n^o}(\tilde{c})S_{n^e}(c)|-q\rangle$$

$$c_{-n} \sim p_n = \sum_i x_i^n$$

$$b_0|q\rangle = q|q\rangle$$

At arbitrary g there is an interacting term with triangular structure:

$$I_3^+(g) = \mathcal{I}_3^+(c; g) + \mathcal{I}_3^+(\tilde{c}; g) + (\sqrt{2}gb_0 + g - 1)(L_0(c) - L_0(\tilde{c})) + 2(1 - g) \sum_{m>0} mc_{-m}\tilde{c}_m$$

also

[Maulik, Okounkov, unpublished]

[Shou, Wu, Yu, 11]



additive spectrum! : $E_{3; n^o, n^e}^\pm(g) = e_{3, n^o}^\pm(g) + e_{3, n^e}^\pm(g) \pm (g - 1)(|n^o| - |n^e|)$

triangular structure: $|n^o, n^e; q\rangle = J_{n^o}^{1/g}(c) J_{n^e}^{1/g}(\tilde{c}) |q\rangle + \dots$

- **example:** states at level 2 in the identity module (3 states out of 5 by symmetry reasons)

$$|n^o, n^e; q\rangle = |n^e, n^o; -q\rangle \quad \rightarrow \quad |n^o, n^e; 0\rangle = |n^e, n^o; 0\rangle$$

$$\begin{aligned} |[2], [\emptyset] > &= \left[(2 - 3g)a_{-2} + \left(\frac{3}{2} - \frac{1}{g} \right) \sqrt{2g}a_{-1}^2 + \sqrt{2g}L_{-2} \right] |N > \\ |[1, 1], [\emptyset] > &= \left[(3 - 2g)a_{-2} + \left(\frac{3}{2} - g \right) \sqrt{2g}a_{-1}^2 + \sqrt{2g}L_{-2} \right] |N > \\ |[1], [1] > &= \left[(1 - g)a_{-2} - \frac{1}{2} \sqrt{2g}a_{-1}^2 + \sqrt{2g}L_{-2} \right] |N > \end{aligned}$$

WA_{k-1} algebras

The same construction extends to WA_{k-1} algebras

k-1 bosons x u(1) component: $\longrightarrow$ k- component CS Hamiltonian

$$I_3^{(\pm)}(g) = 2(1-g) \sum_{m \geq 1} m a_{-m} a_m \pm \sqrt{2g} \sum_{m \geq 1} (a_{-m} L_m + L_{-m} a_m) \pm \sqrt{\frac{g}{2}} \left(\sum_{m,k \geq 1} a_{-m-k} a_m a_k + a_{-m} a_{-k} a_{m+k} \right) \pm \widetilde{W}_0$$

c_m^j : k copies of mutually commuting bosons

$$I_3^+(g) = \sum_{j=1}^k \mathcal{I}^\pm(c^j; g) + 2(1-g) \sum_{j < l} \sum_{m \geq 1} m : c_{-m}^j c_m^l : + (1-g) \sum_j d_j L_0(c^j) + \text{zero modes}$$

$\longrightarrow$ additive spectrum depending on **k partitions** ($\sim$ AGT conjecture for U(k) theories)

Conclusions

- We have learned how to characterize the states of the $\text{Vir} \times H$ CFT, or $WA_{k-1} \times H$, in terms of CS integrals of motion
- AFLT: this basis gives an efficient way to compute matrix elements of the fields (representation of the conformal blocks)
- Similar structure in the FQHE (different physics)

open questions

- Theory of non-polynomial CS eigenfunctions?
- How to systematically generate the integrals of motion (transfer matrix?) in CFT?
[Maulik, Okounkov, unpublished] in 4d gauge theory context
- Relation with the integrable structure uncovered by [Bazhanov, Lukyanov, Zamolodchikov, 94-98] (no Heisenberg factor)?

...