Modular bootstrap in Liouville field theory

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work with L. Hadasz and Z. Jaskolski

Virasoro algebra

$$[L_n, L_m] \quad = \quad (n-m)\ L_{n+m} + \frac{c}{12} n(n^2-1)\ \delta_{n+m,0}$$

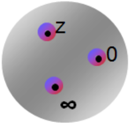
Verma module $\mathcal{V}_\Delta$

$$L_0 \nu_\Delta = \Delta \nu_\Delta, \qquad L_m \nu_\Delta = 0, \quad (m > 0),$$

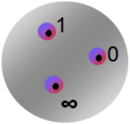
$$\xi_\Delta = L_N \nu_\Delta = L_{n_1} \dots L_{n_j} \nu_\Delta, \ (n_i < 0)$$

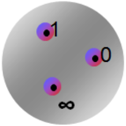
space of states	1:1	space of fields
$\mathcal{H} = \bigoplus_{(\Delta,\bar{\Delta})} \mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_{\bar{\Delta}}$	$\Leftrightarrow$	$\bigoplus_{(\Delta,\bar{\Delta})} \left\{ \phi_{\Delta,\bar{\Delta}} \right\}$
$\lvert \nu_\Delta \otimes \bar{\nu}_{\bar{\Delta}} \rangle$ - h. w. states	$\Leftrightarrow$	primary fields $\phi_{\Delta,\bar{\Delta}}(z,\bar{z})$
$\lvert L_N \nu_\Delta \otimes \bar{L}_M \bar{\nu}_{\bar{\Delta}} \rangle$ - excited states	$\Leftrightarrow$	descendant fields: $\mathcal{L}_N \bar{\mathcal{L}}_M \phi_{\Delta,\bar{\Delta}}(z,\bar{z})$

3-point blocks

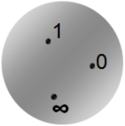


$$= z^{\Delta(\xi_3)-\Delta(\xi_2)-\Delta(\xi_1)}\,\bar{z}^{\bar{\Delta}(\bar{\xi}_3)-\bar{\Delta}(\bar{\xi}_2)-\bar{\Delta}(\bar{\xi}_1)} \times$$





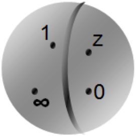
$$= \rho(\xi_3, \xi_2, \xi_1)\,\rho(\bar{\xi}_3, \bar{\xi}_2, \bar{\xi}_1) \times$$



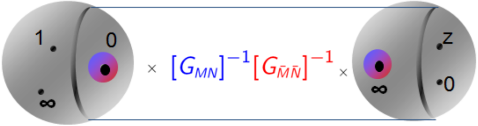
structure constant C_{321}

- factorization onto holomorphic and antiholomorphic sectors
- **3-point blocks** determined by the Ward identities up to one constant

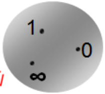
4-point blocks



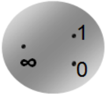
$$= \sum_{(\Delta,\bar\Delta)} \sum_{M,N,\bar M,\bar N}$$



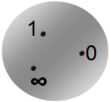
$$= \sum_{(\Delta,\bar\Delta)} \sum_{M,N,\bar M,\bar N}$$



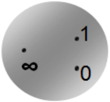
$$\times\ \rho_M \rho_{\bar M} \, [G_{MN}]^{-1} [G_{\bar M\bar N}]^{-1} \, z^{|N|-\Delta_{12}} \, \bar z^{|\bar N|-\bar\Delta_{12}} \rho_N \, \rho_{\bar N} \times$$



$$= \sum_{(\Delta,\bar\Delta)} \mathcal{F}_\Delta \begin{bmatrix} \Delta_4 & \Delta_1 \\ \Delta_3 & \Delta_2 \end{bmatrix} (z) \, \mathcal{F}_{\bar\Delta} \begin{bmatrix} \bar\Delta_4 & \bar\Delta_1 \\ \bar\Delta_3 & \bar\Delta_2 \end{bmatrix} (\bar z) \times$$



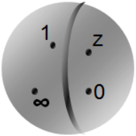
$\times$



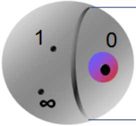
$$G_{MN} = \langle L_M \nu_\Delta | L_N \nu_\Delta \rangle$$

$$\mathcal{F}_\Delta \begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} (z) = \sum_{n=0}^{\infty} z^{\Delta-\Delta_1-\Delta_2+n} \sum_{|M|=|N|=n} \rho_M \, [G_{MN}]^{-1} \, \rho_N$$

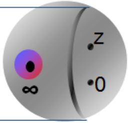
4-point blocks



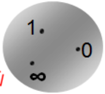
$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}}$$



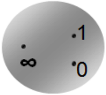
$$\times \left[G_{MN}\right]^{-1}\left[G_{\bar{M}\bar{N}}\right]^{-1} \times$$



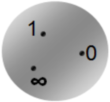
$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}}$$



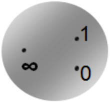
$$\times \rho_M \rho_{\bar{M}} \left[G_{MN}\right]^{-1}\left[G_{\bar{M}\bar{N}}\right]^{-1} z^{|N|-\Delta_{12}} \, \bar{z}^{|\bar{N}|-\bar{\Delta}_{12}} \rho_N \, \rho_{\bar{N}} \times$$



$$= \sum_{(\Delta,\bar{\Delta})} \mathcal{F}_{\Delta}\begin{bmatrix} \Delta_4 & \Delta_1 \\ \Delta_3 & \Delta_2 \end{bmatrix}(z) \, \mathcal{F}_{\bar{\Delta}}\begin{bmatrix} \bar{\Delta}_4 & \bar{\Delta}_1 \\ \bar{\Delta}_3 & \bar{\Delta}_2 \end{bmatrix}(\bar{z}) \times$$



$$\times$$



$$G_{MN} = \langle L_M \nu_\Delta | L_N \nu_\Delta \rangle$$

$$\mathcal{F}_{\Delta}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(z) = \sum_{n=0}^{\infty} z^{\Delta-\Delta_1-\Delta_2+n} \sum_{|M|=|N|=n} \rho_M \left[G_{MN}\right]^{-1} \rho_N$$

4-point conformal block

$$\mathcal{F}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}(z) = z^{\Delta-\Delta_2-\Delta_1}\left(1+\sum_{n\in\mathbb{N}} z^n F^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}\right),$$

$$F^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \sum_{n=|M|=|N|} \rho(\nu_4,\nu_3,L_M\nu_\Delta)\left[G^n_{c,\Delta_p}\right]^{-1}_{MN}\rho(L_N\nu_\Delta,\nu_2,\nu_1)$$

$$F^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \mathrm{h}^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} + \sum_{1\leq rs\leq n}\frac{\mathcal{R}^n_{c,\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}}{\Delta-\Delta_{rs}(c)} = \mathrm{f}^n_{\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} + \sum_{1<rs\leq n}\frac{\widetilde{\mathcal{R}}^n_{\Delta,\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}}{c-c_{rs}(\Delta)}$$

$$\widetilde{\mathcal{R}}^n_{\Delta,\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = -\frac{\partial c_{rs}(\Delta)}{\partial\Delta}\mathcal{R}^n_{c_{rs}(\Delta),\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix},$$

$$\frac{\partial c_{rs}(\Delta)}{\partial\Delta} = 4\,\frac{c_{rs}(\Delta)-1}{(r^2-1)\,b^4_{rs}(\Delta)-(s^2-1)}.$$

4-point conformal block

$$\mathcal{F}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}(z) = z^{\Delta-\Delta_2-\Delta_1}\left(1+\sum_{n\in\mathbb{N}} z^n F^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}\right),$$

$$F^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \sum_{n=|M|=|N|}\rho(\nu_4,\nu_3,L_M\nu_\Delta)\left[G^n_{c,\Delta_p}\right]^{-1}_{MN}\rho(L_N\nu_\Delta,\nu_2,\nu_1)$$

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$$\widetilde{\mathcal{R}}^n_{\Delta,\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = -\frac{\partial c_{rs}(\Delta)}{\partial\Delta}\,\mathcal{R}^n_{c_{rs}(\Delta),\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix},$$

$$\frac{\partial c_{rs}(\Delta)}{\partial\Delta} = 4\,\frac{c_{rs}(\Delta)-1}{(r^2-1)\,b^4_{rs}(\Delta)-(s^2-1)}.$$

residue at $\Delta \to \Delta_{rs}$

$$\mathcal{R}^n_{c,\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \lim_{\Delta\to\Delta_{rs}} (\Delta - \Delta_{rs}(c)) F^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

$$= A_{rs}(c) P^{rs}_c\begin{bmatrix}\Delta_3\\ \Delta_4\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix} F^{n-rs}_{c,\Delta_{rs}+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

$$A_{rs}(c) = \lim_{\Delta\to\Delta_{rs}}\left(\frac{\langle\chi^\Delta_{rs}|\chi^\Delta_{rs}\rangle}{\Delta - \Delta_{rs}(c)}\right)^{-1} = \frac{1}{2}\prod_{p=1-r}^{r}\prod_{\substack{q=1-s\\(p,q)\neq(0,0),(r,s)}}^{s}\frac{1}{pb+qb^{-1}}$$

$$P^{rs}_c\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix} = \rho(\chi_{rs},\nu_2,\nu_1) = (-1)^{rs} Y_{rs}\left(\tfrac{\lambda_1+\lambda_2}{2}\right) Y_{rs}\left(\tfrac{\lambda_1-\lambda_2}{2}\right)$$

$$Y_{rs}(\mu) \equiv \prod_{\substack{p=1-r\\ p+r=1 \bmod 2}}^{r-1}\ \prod_{\substack{q=1-s\\ q+s=1 \bmod 2}}^{s-1}\left(\mu - \frac{pb+qb^{-1}}{2}\right),\quad \Delta_i = \frac{1}{4}(Q^2 - \lambda_i^2)$$

residue at $\Delta \to \Delta_{rs}$

$$\mathcal{R}^n_{c,\,rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \lim_{\Delta\to\Delta_{rs}}(\Delta-\Delta_{rs}(c))F^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

$$= A_{rs}(c)P^{rs}_c\begin{bmatrix}\Delta_3\\ \Delta_4\end{bmatrix}P^{rs}_c\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}F^{n-rs}_{c,\Delta_{rs}+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

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recursive relation

$$\mathcal{F}^{n}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \mathrm{f}^{n}_{\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} - \sum_{1\le rs\le n}\frac{\frac{\partial c_{rs}(\Delta)}{\partial\Delta}A_{rs}(c_{rs}(\Delta))\,P^{rs}_{c_{rs}(\Delta)}\begin{bmatrix}\Delta_3\\ \Delta_4\end{bmatrix}P^{rs}_{c_{rs}(\Delta)}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}}{c-c_{rs}(\Delta)}\,\mathcal{F}^{n-rs}_{c_{rs}(\Delta),\Delta+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

- c-independent term can be calculated as a $c\to\infty$ limit of the block

$$\mathrm{f}^{n}_{\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \frac{1}{n!}\frac{(\Delta+\Delta_3-\Delta_4)_n(\Delta+\Delta_2-\Delta_1)_n}{(2\Delta)_n},$$

$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol.

$$\mathcal{F}^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \mathrm{h}^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} + \sum_{1\leq rs\leq n}\frac{A_{rs}(c)\, P^{rs}_c\begin{bmatrix}\Delta_3\\ \Delta_4\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}}{\Delta-\Delta_{rs}(c)}\, \mathcal{F}^{n-rs}_{c,\Delta_{rs}+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

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$$\begin{aligned}\mathcal{F}^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} &= \mathrm{f}^n_{\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}\\ &- \sum_{1\leq rs\leq n}\frac{\frac{\partial c_{rs}(\Delta)}{\partial\Delta}A_{rs}(c_{rs}(\Delta))\, P^{rs}_{c_{rs}(\Delta)}\begin{bmatrix}\Delta_3\\ \Delta_4\end{bmatrix} P^{rs}_{c_{rs}(\Delta)}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}}{c-c_{rs}(\Delta)}\, \mathcal{F}^{n-rs}_{c_{rs}(\Delta),\Delta+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}\end{aligned}$$

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Elliptic block

large Δ asymptotic $\Rightarrow$ **regular prefactor**:

$$\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) = (16q)^{\Delta-\frac{c-1}{24}}\, z^{\frac{c-1}{24}-\Delta_1-\Delta_2}\,(1-z)^{\frac{c-1}{24}-\Delta_2-\Delta_3} \times (\theta_3(q))^{\frac{c-1}{2}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\,\mathcal{H}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q),$$

elliptic nome & its inverse

$$q(z) = e^{i\pi\tau}, \qquad \tau(z) = i\frac{K(1-z)}{K(z)}, \qquad z(q) = \frac{\theta_2^4(q)}{\theta_3^4(q)}$$

$K(z)$ - complete elliptic integral of the first kind

- the only singularities of the conformal block: branching points at $0, 1, \infty$
- the singularity at 0 - exported to the multiplicative factor
- $z(q)$ - universal covering of $\mathbb{C}\setminus\{1\}$ by the Poincare disc
- the **elliptic block** is a power series in nome

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elliptic nome & its inverse

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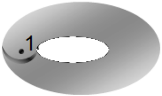
Elliptic recursion

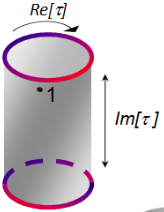
$$\mathcal{H}\begin{bmatrix}\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1\end{bmatrix}(q) = \sum_n (16q)^n H^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1\end{bmatrix}$$

$$H^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1\end{bmatrix} = \delta_{n,0} + \sum_{r,s>0} \frac{A_{rs}(c)\, P^{rs}_c\begin{bmatrix}\Delta_3 \\ \Delta_4\end{bmatrix}\, P^{rs}_c\begin{bmatrix}\Delta_2 \\ \Delta_1\end{bmatrix}}{\Delta - \Delta_{rs}} H^{n-rs}_{c,\Delta_{rs}+rs}\begin{bmatrix}\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1\end{bmatrix}$$

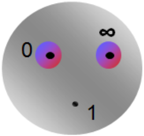
1. 4-point conformal blocks on a sphere
2. 1-point block on a torus
3. Modular invariance
   - toric 1-pt block vs spheric 4-point blocks
   - modular invariance in the Liouville theory



$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}} [G_{MN}]^{-1} [G_{\bar{M}\bar{N}}]^{-1} \times$$

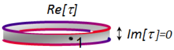


$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}} q^{\Delta - \frac{c}{24} + |N|} \bar{q}^{\bar{\Delta} - \frac{c}{24} + |\bar{N}|} [G_{MN}]^{-1} [G_{\bar{M}\bar{N}}]^{-1} \times$$



$$\langle \phi_{\lambda,\bar{\lambda}} \rangle = \mathrm{Tr}\left(e^{-(Im\tau)\hat{H} + i(Re\tau)\hat{P}} \phi_\lambda(1,1) \right) = (q\bar{q})^{-\frac{c}{24}} \mathrm{Tr}\left(q^{L_0} \bar{q}^{\bar{L}_0} \phi_\lambda(1,1) \right)$$

$$\hat{H} = 2\pi(L_0 + \bar{L}_0) - \frac{\pi c}{6}, \qquad \hat{P} = 2\pi(L_0 - \bar{L}_0), \qquad q = e^{2\pi i \tau}$$



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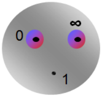
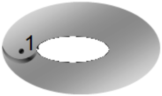
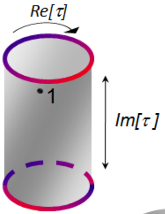
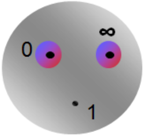




$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}} [G_{MN}]^{-1} [G_{\bar{M}\bar{N}}]^{-1} \times$$

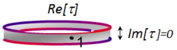


$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}} q^{\Delta - \frac{c}{24} + |N|} \bar{q}^{\bar{\Delta} - \frac{c}{24} + |\bar{N}|} [G_{MN}]^{-1} [G_{\bar{M}\bar{N}}]^{-1} \times$$



$$\langle \phi_{\lambda,\bar{\lambda}} \rangle = \mathrm{Tr}\left(e^{-(Im\tau)\hat{H} + i(Re\tau)\hat{P}} \phi_\lambda(1,1) \right) = (q\bar{q})^{-\frac{c}{24}} \mathrm{Tr}\left(q^{L_0} \bar{q}^{\bar{L}_0} \phi_\lambda(1,1) \right)$$

$$\hat{H} = 2\pi(L_0 + \bar{L}_0) - \frac{\pi c}{6}, \qquad \hat{P} = 2\pi(L_0 - \bar{L}_0), \qquad q = e^{2\pi i \tau}$$



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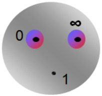
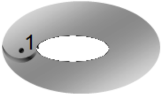
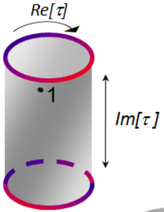
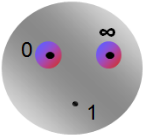




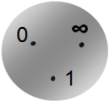
$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}} [G_{MN}]^{-1} [G_{\bar{M}\bar{N}}]^{-1} \times$$



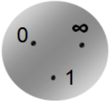
$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}} q^{\Delta - \frac{c}{24} + |N|} \bar{q}^{\bar{\Delta} - \frac{c}{24} + |\bar{N}|} [G_{MN}]^{-1} [G_{\bar{M}\bar{N}}]^{-1} \times$$



$$= \sum_{(\Delta,\bar{\Delta})} \sum_{M,N,\bar{M},\bar{N}} q^{\Delta - \frac{c}{24} + |N|} \bar{q}^{\bar{\Delta} - \frac{c}{24} + |\bar{N}|} \times [G_{MN}]^{-1} [G_{\bar{M}\bar{N}}]^{-1} \times \rho_{MN}\, \rho_{\bar{M}\bar{N}} \times$$



$$= \sum_{(\Delta,\bar{\Delta})} \mathcal{F}^{\lambda}_{\Delta}(q)\, \mathcal{F}^{\bar{\lambda}}_{\bar{\Delta}}(\bar{q}) \times$$



1-point conformal block on a torus

$$\mathcal{F}^{\lambda}_{c,\Delta}(q) = q^{\Delta-\frac{c}{24}} \sum_{n=0}^{\infty} q^n F^{\lambda,n}_{c,\Delta},$$

$$F^{\lambda,n}_{c,\Delta} = \sum_{n=|M|=|N|} \rho(L_N\nu_\Delta, \nu_\lambda, L_M\nu_\Delta) \left[G^n_{c,\Delta}\right]^{-1}_{MN}$$

- coefficients expressed by sums over simple poles in Δ:

$$F^{\lambda,n}_{c,\Delta} = \mathrm{h}^{\lambda,n}_c + \sum_{1\leq rs\leq n} \frac{\mathcal{R}^{\lambda,n}_{c,\,rs}}{\Delta - \Delta_{rs}(c)}$$

residue at Δ_{rs}

$$\mathcal{R}^{\lambda,n}_{c,\,rs} = \lim_{\Delta\to\Delta_{rs}} (\Delta - \Delta_{rs}(c))\, F^{\lambda,n}_{c,\Delta} = A_{rs}(c) P^{rs}_c\begin{bmatrix} \Delta_\lambda \\ \Delta_{rs}+rs \end{bmatrix} P^{rs}_c\begin{bmatrix} \Delta_\lambda \\ \Delta_{rs} \end{bmatrix} F^{\lambda,n-rs}_{c,\Delta_{rs}+rs}.$$

$$\rho(\chi_{rs}, \nu_\lambda, \chi_{rs}) = \rho(\chi_{rs}, \nu_\lambda, \nu_{\Delta_{rs}+rs})\ \rho(\nu_{\Delta_{rs}}, \nu_\lambda, \chi_{rs}) = P^{rs}_c\begin{bmatrix} \Delta_\lambda \\ \Delta_{rs}+rs \end{bmatrix} P^{rs}_c\begin{bmatrix} \Delta_\lambda \\ \Delta_{rs} \end{bmatrix}$$

1-point conformal block on a torus

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$$F^{\lambda,n}_{c,\Delta} = \mathrm{h}^{\lambda,n}_c + \sum_{1\leq rs\leq n} \frac{\mathcal{R}^{\lambda,n}_{c,\,rs}}{\Delta - \Delta_{rs}(c)}$$

residue at Δ_{rs}

$$\mathcal{R}^{\lambda,n}_{c,\,rs} = \lim_{\Delta\to\Delta_{rs}} (\Delta-\Delta_{rs}(c))\, F^{\lambda,n}_{c,\Delta} = A_{rs}(c) P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}+rs\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}\end{bmatrix} F^{\lambda,n-rs}_{c,\Delta_{rs}+rs}.$$

$$\rho(\chi_{rs},\nu_\lambda,\chi_{rs}) = \rho(\chi_{rs},\nu_\lambda,\nu_{\Delta_{rs}+rs})\ \rho(\nu_{\Delta_{rs}},\nu_\lambda,\chi_{rs}) = P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}+rs\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}\end{bmatrix}$$

1-point conformal block on a torus

$$\mathcal{F}^{\lambda}_{c,\Delta}(q) = q^{\Delta-\frac{c}{24}} \sum_{n=0}^{\infty} q^n F^{\lambda,n}_{c,\Delta},$$

$$F^{\lambda,n}_{c,\Delta} = \sum_{n=|M|=|N|} \rho(L_N\nu_\Delta, \nu_\lambda, L_M\nu_\Delta) \left[G^n_{c,\Delta}\right]^{-1}_{MN}$$

- coefficients expressed by sums over simple poles in Δ:

$$F^{\lambda,n}_{c,\Delta} = \mathrm{h}^{\lambda,n}_c + \sum_{1\leq rs\leq n} \frac{\mathcal{R}^{\lambda,n}_{c,\,rs}}{\Delta - \Delta_{rs}(c)}$$

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$$\mathcal{R}^{\lambda,n}_{c,\,rs} = \lim_{\Delta\to\Delta_{rs}} (\Delta-\Delta_{rs}(c))\, F^{\lambda,n}_{c,\Delta} = A_{rs}(c) P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}+rs\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}\end{bmatrix} F^{\lambda,n-rs}_{c,\Delta_{rs}+rs}.$$

$$\rho(\chi_{rs}, \nu_\lambda, \chi_{rs}) = \rho(\chi_{rs}, \nu_\lambda, \nu_{\Delta_{rs}+rs})\ \rho(\nu_{\Delta_{rs}}, \nu_\lambda, \chi_{rs}) = P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}+rs\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}\end{bmatrix}$$

Large Δ asymptotic

$$\lim_{\Delta\to\infty}\sum q^n \mathrm{h}_c^{\lambda,n} = \lim_{\Delta\to\infty}\sum q^n F_{c,\Delta}^{\lambda,n} = \prod_{n=1}^{\infty}(1-q^n)^{-1} = q^{\frac{1}{24}}\eta(q)^{-1}$$

$\eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ - Dedekind eta function

elliptic 1-point block on a torus

$$\mathcal{F}_{c,\Delta}^{\lambda}(q) = q^{\frac{c}{24}-\Delta}\, q^{\frac{1}{24}}\,(\eta(q))^{-1}\,\mathcal{H}_{c,\Delta}^{\lambda}(q), \qquad \mathcal{H}_{c,\Delta}^{\lambda}(q) = \sum_{n=0}^{\infty} q^n H_{c,\Delta}^{\lambda,n}$$

recursion relation

$$H_{c,\Delta}^{\lambda,n} = \delta_{n,0} + \sum_{1\leq rs\leq n}\frac{A_{rs}(c)\,P_c^{rs}\left[{}^{\Delta_\lambda}_{\Delta_{rs}+rs}\right]\,P_c^{rs}\left[{}^{\Delta_\lambda}_{\Delta_{rs}}\right]}{\Delta-\Delta_{rs}(c)}\,H_{c,\Delta_{rs}+rs}^{\lambda,n-rs}$$

Large Δ asymptotic

$$\lim_{\Delta\to\infty}\sum q^n \mathrm{h}_c^{\lambda,n} = \lim_{\Delta\to\infty}\sum q^n F_{c,\Delta}^{\lambda,n} = \prod_{n=1}^{\infty}(1-q^n)^{-1} = q^{\frac{1}{24}}\eta(q)^{-1}$$

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elliptic 1-point block on a torus

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$$H_{c,\Delta}^{\lambda,n} = \delta_{n,0} + \sum_{1\leq rs\leq n}\frac{A_{rs}(c)\,P_c^{rs}\left[{}^{\Delta_\lambda}_{\Delta_{rs}+rs}\right]P_c^{rs}\left[{}^{\Delta_\lambda}_{\Delta_{rs}}\right]}{\Delta-\Delta_{rs}(c)}\,H_{c,\Delta_{rs}+rs}^{\lambda,n-rs}$$

Large Δ asymptotic

$$\lim_{\Delta\to\infty}\sum q^n \mathrm{h}_c^{\lambda,n} = \lim_{\Delta\to\infty}\sum q^n F_{c,\Delta}^{\lambda,n} = \prod_{n=1}^{\infty}(1-q^n)^{-1} = q^{\frac{1}{24}}\eta(q)^{-1}$$

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elliptic 1-point block on a torus

$$\mathcal{F}_{c,\Delta}^{\lambda}(q) = q^{\frac{c}{24}-\Delta}\, q^{\frac{1}{24}}\,(\eta(q))^{-1}\,\mathcal{H}_{c,\Delta}^{\lambda}(q), \qquad \mathcal{H}_{c,\Delta}^{\lambda}(q) = \sum_{n=0}^{\infty} q^n H_{c,\Delta}^{\lambda,n}$$

recursion relation

$$H_{c,\Delta}^{\lambda,n} \quad = \quad \delta_{n,0} + \sum_{1\leq rs\leq n} \frac{A_{rs}(c)\, P_c^{rs}\left[{}^{\Delta_\lambda}_{\Delta_{rs}+rs}\right] P_c^{rs}\left[{}^{\Delta_\lambda}_{\Delta_{rs}}\right]}{\Delta-\Delta_{rs}(c)}\, H_{c,\Delta_{rs}+rs}^{\lambda,n-rs}$$

elliptic recursion relations for 4-point blocks

$$H^{n}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \delta_{n,0} + \sum_{1\leq rs\leq n} \frac{A_{rs}(c)\, P^{rs}_{c}\begin{bmatrix}\Delta_3\\ \Delta_4\end{bmatrix} P^{rs}_{c}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}}{\Delta - \Delta_{rs}} H^{n-rs}_{c,\Delta_{rs}+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

recursion relation for 1-point toric blocks

$$H^{\lambda,n}_{c,\Delta} = \delta_{n,0} + \sum_{1\leq rs\leq n} \frac{A_{rs}(c)\, P^{rs}_{c}\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}+rs\end{bmatrix} P^{rs}_{c}\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}\end{bmatrix}}{\Delta - \Delta_{rs}(c)} H^{\lambda,n-rs}_{c,\Delta_{rs}+rs}$$

elliptic recursion relations for 4-point blocks

$$H^n_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} = \delta_{n,0} + \sum_{1\le rs\le n} \frac{A_{rs}(c)\, P^{rs}_c\begin{bmatrix}\Delta_3\\ \Delta_4\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}}{\Delta - \Delta_{rs}} H^{n-rs}_{c,\Delta_{rs}+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}$$

recursion relation for 1-point toric blocks

$$H^{\lambda,n}_{c,\Delta} = \delta_{n,0} + \sum_{1\le rs\le n} \frac{A_{rs}(c)\, P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}+rs\end{bmatrix} P^{rs}_c\begin{bmatrix}\Delta_\lambda\\ \Delta_{rs}\end{bmatrix}}{\Delta - \Delta_{rs}(c)} H^{\lambda,n-rs}_{c,\Delta_{rs}+rs}$$

I Poghosian identities

$$
\begin{aligned}
\mathcal{H}^{\lambda}_{c,\Delta}(q) &= \mathcal{H}_{c,\Delta}\begin{bmatrix} \frac{1}{2}\lambda & \frac{1}{2}\lambda \\ \frac{1}{2}(b-\frac{1}{b}) & \frac{1}{2}(b+\frac{1}{b}) \end{bmatrix}(q) \\
&= \mathcal{H}_{c,\Delta}\begin{bmatrix} \frac{1}{2}(\lambda-b) & \frac{1}{2}(\lambda+b) \\ \frac{1}{2b} & \frac{1}{2b} \end{bmatrix}(q) = \mathcal{H}_{c,\Delta}\begin{bmatrix} \frac{1}{2}(\lambda-\frac{1}{b}) & \frac{1}{2}(\lambda+\frac{1}{b}) \\ \frac{1}{2}b & \frac{1}{2}b \end{bmatrix}(q)
\end{aligned}
$$

$$
P^{rs}_{c}\begin{bmatrix} \Delta_\lambda \\ \Delta_{rs}+rs \end{bmatrix} P^{rs}_{c}\begin{bmatrix} \Delta_\lambda \\ \Delta_{rs} \end{bmatrix} = (16)^{rs}\, P^{rs}_{c}\begin{bmatrix} \Delta_2 \\ \Delta_1 \end{bmatrix} \; P^{rs}_{c}\begin{bmatrix} \Delta_3 \\ \Delta_4 \end{bmatrix}
$$

for

$$
\begin{aligned}
(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \left(\tfrac{Q}{2},\tfrac{\lambda}{2},\tfrac{\lambda}{2},\tfrac{b}{2}-\tfrac{1}{2b}\right) \\
(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \left(\tfrac{1}{2b},\tfrac{\lambda}{2}+\tfrac{b}{2},\tfrac{\lambda}{2}-\tfrac{b}{2},\tfrac{1}{2b}\right) \\
(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \left(\tfrac{b}{2},\tfrac{\lambda}{2}+\tfrac{1}{2b},\tfrac{\lambda}{2}-\tfrac{1}{2b},\tfrac{b}{2}\right)
\end{aligned}
$$

I Poghosian identities

$$\begin{aligned}
\mathcal{H}^{\lambda}_{c,\Delta}(q) &= \mathcal{H}_{c,\Delta}\left[\begin{matrix}\frac{1}{2}\lambda & \frac{1}{2}\lambda \\ \frac{1}{2}(b-\frac{1}{b}) & \frac{1}{2}(b+\frac{1}{b})\end{matrix}\right](q) \\
&= \mathcal{H}_{c,\Delta}\left[\begin{matrix}\frac{1}{2}(\lambda-b) & \frac{1}{2}(\lambda+b) \\ \frac{1}{2b} & \frac{1}{2b}\end{matrix}\right](q) = \mathcal{H}_{c,\Delta}\left[\begin{matrix}\frac{1}{2}(\lambda-\frac{1}{b}) & \frac{1}{2}(\lambda+\frac{1}{b}) \\ \frac{1}{2}b & \frac{1}{2}b\end{matrix}\right](q)
\end{aligned}$$

$$P^{rs}_{c}\left[\begin{matrix}\Delta_\lambda \\ \Delta_{rs}+rs\end{matrix}\right] P^{rs}_{c}\left[\begin{matrix}\Delta_\lambda \\ \Delta_{rs}\end{matrix}\right] = (16)^{rs}\, P^{rs}_{c}\left[\begin{matrix}\Delta_2 \\ \Delta_1\end{matrix}\right]\; P^{rs}_{c}\left[\begin{matrix}\Delta_3 \\ \Delta_4\end{matrix}\right]$$

for

$$\begin{aligned}
(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \left(\tfrac{Q}{2},\tfrac{\lambda}{2},\tfrac{\lambda}{2},\tfrac{b}{2}-\tfrac{1}{2b}\right) \\
(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \left(\tfrac{1}{2b},\tfrac{\lambda}{2}+\tfrac{b}{2},\tfrac{\lambda}{2}-\tfrac{b}{2},\tfrac{1}{2b}\right) \\
(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \left(\tfrac{b}{2},\tfrac{\lambda}{2}+\tfrac{1}{2b},\tfrac{\lambda}{2}-\tfrac{1}{2b},\tfrac{b}{2}\right)
\end{aligned}$$

II A Poghosian identity

$$\mathcal{H}_{c,\Delta}\begin{bmatrix}\frac{1}{2b} & \mu \\ \frac{1}{2b} & \frac{1}{2b}\end{bmatrix}(q) = \mathcal{H}_{c',\Delta'}\begin{bmatrix}\frac{1}{\sqrt{2}}\mu & \frac{1}{\sqrt{2}}\mu \\ \frac{1}{2}(b'-\frac{1}{b'}) & \frac{1}{2}(b'+\frac{1}{b'})\end{bmatrix}(q^2)$$

$$= \mathcal{H}_{c',\Delta'}\begin{bmatrix}\frac{1}{\sqrt{2}}\mu-\frac{1}{2}b' & \frac{1}{\sqrt{2}}\mu+\frac{1}{2}b' \\ \frac{1}{2b'} & \frac{1}{2b'}\end{bmatrix}(q^2)$$

$$= \mathcal{H}_{c',\Delta'}\begin{bmatrix}\frac{1}{\sqrt{2}}\mu-\frac{1}{2b'} & \frac{1}{\sqrt{2}}\mu+\frac{1}{2b'} \\ \frac{1}{2}b' & \frac{1}{2}b'\end{bmatrix}(q^2)$$

$$c' = 1+6\left(b'+\tfrac{1}{b'}\right)^2 \qquad b' = \sqrt{2}b$$

$$\Delta'_{\lambda'} = \frac{(b'+\frac{1}{b'})^2}{4} - \frac{(\lambda')^2}{4} \qquad \lambda' = \frac{\lambda}{\sqrt{2}}$$

$$\frac{R_c^{2k,s}\begin{bmatrix}\frac{1}{2b} & \mu \\ \frac{1}{2b} & \frac{1}{2b}\end{bmatrix}}{\Delta-\Delta_{rs}} = 16^{-ks}\frac{R_{c'}^{k,s}\begin{bmatrix}\lambda'_3 & \lambda'_2 \\ \lambda'_4 & \lambda'_1\end{bmatrix}}{(\Delta'-\Delta'_{rs})}$$

II A Poghosian identity

$$\begin{aligned}
\mathcal{H}_{c,\Delta}\begin{bmatrix} \frac{1}{2b} & \mu \\ \frac{1}{2b} & \frac{1}{2b} \end{bmatrix}(q) &= \mathcal{H}_{c',\Delta'}\begin{bmatrix} \frac{1}{\sqrt{2}}\mu & \frac{1}{\sqrt{2}}\mu \\ \frac{1}{2}(b'-\frac{1}{b'}) & \frac{1}{2}(b'+\frac{1}{b'}) \end{bmatrix}(q^2) \\
&= \mathcal{H}_{c',\Delta'}\begin{bmatrix} \frac{1}{\sqrt{2}}\mu-\frac{1}{2}b' & \frac{1}{\sqrt{2}}\mu+\frac{1}{2}b' \\ \frac{1}{2b'} & \frac{1}{2b'} \end{bmatrix}(q^2) \\
&= \mathcal{H}_{c',\Delta'}\begin{bmatrix} \frac{1}{\sqrt{2}}\mu-\frac{1}{2b'} & \frac{1}{\sqrt{2}}\mu+\frac{1}{2b'} \\ \frac{1}{2}b' & \frac{1}{2}b' \end{bmatrix}(q^2)
\end{aligned}$$

$$c' = 1 + 6\left(b' + \frac{1}{b'}\right)^2 \qquad b' = \sqrt{2}b$$

$$\Delta'_{\lambda'} = \frac{(b'+\frac{1}{b'})^2}{4} - \frac{(\lambda')^2}{4} \qquad \lambda' = \frac{\lambda}{\sqrt{2}}$$

$$\frac{R_c^{2k,s}\begin{bmatrix} \frac{1}{2b} & \mu \\ \frac{1}{2b} & \frac{1}{2b} \end{bmatrix}}{\Delta - \Delta_{rs}} = 16^{-ks}\frac{R_{c'}^{k,s}\begin{bmatrix} \lambda'_3 & \lambda'_2 \\ \lambda'_4 & \lambda'_1 \end{bmatrix}}{(\Delta' - \Delta'_{rs})}$$

II A Poghosian identity

$$
\begin{aligned}
\mathcal{H}_{c,\Delta}\begin{bmatrix} \frac{1}{2b} & \mu \\ \frac{1}{2b} & \frac{1}{2b} \end{bmatrix}(q) &= \mathcal{H}_{c',\Delta'}\begin{bmatrix} \frac{1}{\sqrt{2}}\mu & \frac{1}{\sqrt{2}}\mu \\ \frac{1}{2}(b'-\frac{1}{b'}) & \frac{1}{2}(b'+\frac{1}{b'}) \end{bmatrix}(q^2) \\
&= \mathcal{H}_{c',\Delta'}\begin{bmatrix} \frac{1}{\sqrt{2}}\mu-\frac{1}{2}b' & \frac{1}{\sqrt{2}}\mu+\frac{1}{2}b' \\ \frac{1}{2b'} & \frac{1}{2b'} \end{bmatrix}(q^2) \\
&= \mathcal{H}_{c',\Delta'}\begin{bmatrix} \frac{1}{\sqrt{2}}\mu-\frac{1}{2b'} & \frac{1}{\sqrt{2}}\mu+\frac{1}{2b'} \\ \frac{1}{2}b' & \frac{1}{2}b' \end{bmatrix}(q^2)
\end{aligned}
$$

$$c' = 1 + 6\left(b' + \tfrac{1}{b'}\right)^2 \qquad b' = \sqrt{2}b$$

$$\Delta'_{\lambda'} = \frac{(b'+\frac{1}{b'})^2}{4} - \frac{(\lambda')^2}{4} \qquad \lambda' = \frac{\lambda}{\sqrt{2}}$$

$$\frac{R_c^{2k,s}\begin{bmatrix} \frac{1}{2b} & \mu \\ \frac{1}{2b} & \frac{1}{2b} \end{bmatrix}}{\Delta - \Delta_{rs}} = 16^{-ks}\,\frac{R_{c'}^{k,s}\begin{bmatrix} \lambda'_3 & \lambda'_2 \\ \lambda'_4 & \lambda'_1 \end{bmatrix}}{(\Delta' - \Delta'_{rs})}$$

II B Poghosian identity

$$\mathcal{H}_{c,\Delta}\begin{bmatrix}\frac{b}{2} & \mu \\ \frac{b}{2} & \frac{b}{2}\end{bmatrix}(q) = \mathcal{H}_{c',\Delta'}\begin{bmatrix}\frac{1}{\sqrt{2}}\mu & \frac{1}{\sqrt{2}}\mu \\ \frac{1}{2}(b'-\frac{1}{b'}) & \frac{1}{2}(b'+\frac{1}{b'})\end{bmatrix}(q^2)$$

$$= \mathcal{H}_{c',\Delta'}\begin{bmatrix}\frac{1}{\sqrt{2}}\mu-\frac{1}{2}b' & \frac{1}{\sqrt{2}}\mu+\frac{1}{2}b' \\ \frac{1}{2b'} & \frac{1}{2b'}\end{bmatrix}(q^2)$$

$$= \mathcal{H}_{c',\Delta'}\begin{bmatrix}\frac{1}{\sqrt{2}}\mu-\frac{1}{2b'} & \frac{1}{\sqrt{2}}\mu+\frac{1}{2b'} \\ \frac{1}{2}b' & \frac{1}{2}b'\end{bmatrix}(q^2)$$

$$c' = 1+6\left(b'+\tfrac{1}{b'}\right)^2 \qquad b' = \frac{b}{\sqrt{2}}$$

$$\Delta'_{\lambda'} = \frac{(b'+\frac{1}{b'})^2}{4} - \frac{(\lambda')^2}{4} \qquad \lambda' = \frac{\lambda}{\sqrt{2}}$$

III Poghosian identity

$$\mathcal{H}^{\lambda}_{c,\Delta_\alpha}(q^2) = \mathcal{H}_{c',\Delta'_{\alpha'}}\begin{bmatrix} \frac{1}{2b'} & \frac{\lambda}{\sqrt{2}} \\ \frac{1}{2b'} & \frac{1}{2b'} \end{bmatrix}(q)\,, \quad b' = \frac{b}{\sqrt{2}}\,, \quad \alpha' = \sqrt{2}\alpha\,,$$

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$$c = 1 + 6\left(b + \tfrac{1}{b}\right)^2, \quad \Delta_\alpha = \tfrac{1}{4}\left(b + \tfrac{1}{b}\right)^2 - \tfrac{1}{4}\alpha^2$$

torus

$$q_{\rm torus} = e^{2\pi i \tau_{\rm torus}}$$

4-punctured sphere

$$q_{\rm sphere}(z) = e^{\pi i \tau(z)}\,, \quad \tau(z) = i\frac{K(1-z)}{K(z)}$$

$$q_{\rm torus} = q^2_{\rm sphere}(z)$$

modular transformation

$$\tau_{\rm torus} \to -\frac{1}{\tau_{\rm torus}}$$

crossing transformation

$$z \to 1 - z$$

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1-point function

$$\langle \phi_\lambda \rangle_\tau = \eta(q^2)^{-1} \int\limits_{i\mathbb{R}} \frac{\mathrm{d}\lambda_s}{4i} \left| q^{-\frac{\lambda_s^2}{2}} \mathcal{H}^{\lambda}_{c,\Delta_s}(q^2) \right|^2 C_c\left(-\lambda_s, \lambda, \lambda_s\right),$$

$$\mathcal{H}^{\lambda}_{c,\Delta_\alpha}\left(q^2\right) = \mathcal{H}_{c',\Delta'_{\alpha'}}\begin{bmatrix} \frac{1}{2b'} & \frac{\lambda}{\sqrt{2}} \\ \frac{1}{2b'} & \frac{1}{2b'} \end{bmatrix}(q), \quad b' = \frac{b}{\sqrt{2}}, \quad \alpha' = \sqrt{2}\alpha,$$

$$C_c(-\lambda_s, \lambda, \lambda_s) = 16^{-\lambda_s^2} g(\lambda, b)\, C_{c'}\left(-\tfrac{1}{2b'}, \tfrac{1}{2b'}, \sqrt{2}\lambda_s\right) C_{c'}\left(-\sqrt{2}\lambda_s, \tfrac{\lambda}{\sqrt{2}}, \tfrac{1}{2b'}\right)$$

4-point function

$$\left\langle \phi_{\frac{1}{2b'}} \phi_{\frac{1}{2b'}} \phi_{\frac{\lambda}{\sqrt{2}}}(z) \phi_{\frac{1}{2b'}} \right\rangle^{c'} = f(\lambda, q, b)$$

$$\times \int\limits_{i\mathbb{R}} \frac{\mathrm{d}\lambda_s}{4i} \left| (16q)^{-\frac{\lambda_s^2}{2}} \mathcal{H}_{c',\Delta'_{\alpha'}}\begin{bmatrix} \frac{1}{2b'} & \frac{\lambda}{\sqrt{2}} \\ \frac{1}{2b'} & \frac{1}{2b'} \end{bmatrix}(q) \right|^2 C_{c'}\left(\tfrac{-1}{2b'}, \tfrac{1}{2b'}, \sqrt{2}\lambda_s\right) C_{c'}\left(-\sqrt{2}\lambda_s, \tfrac{\lambda}{\sqrt{2}}, \tfrac{1}{2b'}\right)$$

Fateev Litvinov Neveu Onofri

$$\langle \phi_\lambda \rangle_\tau^c = g(\lambda, b) \left(f(\lambda, q, b)\, \eta^2(q^2) \right)^{-1} \left\langle \phi_{\frac{1}{2b'}} \phi_{\frac{1}{2b'}} \phi_{\frac{\lambda}{\sqrt{2}}}(z) \phi_{\frac{1}{2b'}} \right\rangle^{c'}, \quad b' = \frac{b}{\sqrt{2}}$$

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$$C_c(-\lambda_s,\lambda,\lambda_s) = 16^{-\lambda_s^2}g(\lambda,b)\;C_{c'}\left(-\tfrac{1}{2b'},\tfrac{1}{2b'},\sqrt{2}\lambda_s\right)C_{c'}\left(-\sqrt{2}\lambda_s,\tfrac{\lambda}{\sqrt{2}},\tfrac{1}{2b'}\right)$$

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Fateev Litvinov Neveu Onofri

$$\langle\phi_\lambda\rangle^c_\tau = g(\lambda,b)\left(f(\lambda,q,b)\,\eta^2(q^2)\right)^{-1}\left\langle\phi_{\frac{1}{2b'}}\phi_{\frac{1}{2b'}}\phi_{\frac{\lambda}{\sqrt{2}}}(z)\phi_{\frac{1}{2b'}}\right\rangle^{c'},\quad b'=\frac{b}{\sqrt{2}}$$

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Fateev Litvinov Neveu Onofri

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$$\langle \phi_\lambda \rangle^c_\tau = g(\lambda, b) \left(f(\lambda, q, b)\, \eta^2(q^2) \right)^{-1} \left\langle \phi_{\frac{1}{2b'}} \phi_{\frac{1}{2b'}} \phi_{\frac{\lambda}{\sqrt{2}}}(z) \phi_{\frac{1}{2b'}} \right\rangle^{c'} , \quad b' = \frac{b}{\sqrt{2}}$$

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modular transformation

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modular invariance of 1-point function

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Conclusions

Modular invariance of the Liouville 1-point function on a torus

- relation between 1-point blocks on a torus and 4-point blocks on a sphere (based on the **recursive representations**)
- identities for DOZZ structure constants

Other recent applications of the elliptic recursions:

1. the **modular matrix** expressed in terms of the **fusion matrix**
2. **AGT conjecture**
   - for 1-point toric block [Fateev,Litvinov]
   - for irregular blocks (norms of Whittaker states) [Hadasz,Jaskolski,PS]

Open questions:

- Modular invariance in N=1 supersymmetric Liouville theory (known recursive relations for 4-point blocks; braiding matrices)
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