Thermal correlators in integrable models

G. Takács

HAS Theoretical Physics Research Group
Eötvös University, Budapest

In collaboration with: *B. Pozsgay* (Institute for Theoretical Physics, Amsterdam)
I. Szécsényi (Eötvös University)

Talk given at the 8th Bologna Workshop on: CFT and Intergrable Models and their Applications to Quantum Field Theory, Statistical Mechanics and Condensed Matter Physics

September 12 - 15, 2011

Outline of talk

What do we wish to compute and why?

Goal: correlators in $1+1$ dimensional (integrable) QFTs at finite temperature

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_T = \frac{\mathrm{Tr}\ \mathrm{e}^{-RH} \mathcal{O}_1 \ldots \mathcal{O}_n}{\mathrm{Tr}\ \mathrm{e}^{-RH}} \qquad R = \frac{1}{T}$$

What do we wish to compute and why?

Goal: correlators in $1+1$ dimensional (integrable) QFTs at finite temperature

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_T = \frac{\mathrm{Tr}\ \mathrm{e}^{-RH} \mathcal{O}_1 \ldots \mathcal{O}_n}{\mathrm{Tr}\ \mathrm{e}^{-RH}} \qquad R = \frac{1}{T}$$

Why? A large number of condensed matter systems can be described by such models
(impurities, (quasi) one-dimensional conductors, spin chains/ladders)

What is a correlator good for?

$n = 1$: order parameters
$n = 2$: response functions (susceptibility etc)

One-point functions

$$\langle \mathcal{O}(t,x) \rangle_T = \frac{\sum_n \mathrm{e}^{-RE_n} \langle n | \mathcal{O}(0,0) | n \rangle}{\sum_n \mathrm{e}^{-RE_n}}$$

One-point functions

$$\langle \mathcal{O}(t,x) \rangle_T = \frac{\sum_n \mathrm{e}^{-RE_n} \langle n | \mathcal{O}(0,0) | n \rangle}{\sum_n \mathrm{e}^{-RE_n}}$$

Solution is known: LeClair-Mussardo formula (1999)

$$\langle \mathcal{O}(t,x) \rangle_{T,\mu} = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{k=1}^{N} \left(\int \frac{d\theta_k}{2\pi} \frac{1}{1+\mathrm{e}^{\epsilon(\theta_k)}} \right) \mathcal{F}^{\mathcal{O}}_{2N}(\theta_1, \ldots .\theta_N)_{\mathrm{conn}}$$

One-point functions

$$\langle \mathcal{O}(t,x) \rangle_T = \frac{\sum_n \mathrm{e}^{-RE_n} \langle n | \mathcal{O}(0,0) | n \rangle}{\sum_n \mathrm{e}^{-RE_n}}$$

Solution is known: LeClair-Mussardo formula (1999)

$$\langle \mathcal{O}(t,x) \rangle_{T,\mu} = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{k=1}^{N} \left(\int \frac{d\theta_k}{2\pi} \frac{1}{1+\mathrm{e}^{\epsilon(\theta_k)}} \right) \mathcal{F}_{2N}^{\mathcal{O}}(\theta_1, \dots . \theta_N)_{\mathrm{conn}}$$

$$\begin{aligned} \text{TBA: } \epsilon(\theta) &= mR\cosh\theta - \mu - \int \frac{d\theta'}{2\pi} \varphi(\theta-\theta') \log(1+\mathrm{e}^{-\epsilon(\theta')}) \\ \varphi(\theta) &= -i\frac{\partial}{\partial\theta} \log\, S(\theta) \end{aligned}$$

$$\mathcal{F}_{2N}^{\mathcal{O}}(\theta_1, \dots . \theta_N)_{\mathrm{conn}} = \langle \theta_1, \dots . \theta_N | \mathcal{O}(0,0) | \theta_1, \dots . \theta_N \rangle_{\mathrm{connected}}$$

One-point functions

$$\langle \mathcal{O}(t,x) \rangle_T = \frac{\sum_n \mathrm{e}^{-RE_n} \langle n | \mathcal{O}(0,0) | n \rangle}{\sum_n \mathrm{e}^{-RE_n}}$$

Solution is known: LeClair-Mussardo formula (1999)

$$\langle \mathcal{O}(t,x) \rangle_{T,\mu} = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{k=1}^{N} \left(\int \frac{d\theta_k}{2\pi} \frac{1}{1+\mathrm{e}^{\epsilon(\theta_k)}} \right) \mathcal{F}_{2N}^{\mathcal{O}}(\theta_1, \dots .\theta_N)_{\mathrm{conn}}$$

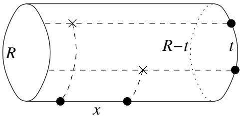
$$\begin{aligned} \text{TBA: } \epsilon(\theta) &= mR\cosh\theta - \mu - \int \frac{d\theta'}{2\pi} \varphi(\theta-\theta') \log(1+\mathrm{e}^{-\epsilon(\theta')}) \\ \varphi(\theta) &= -i\frac{\partial}{\partial\theta} \log\, S(\theta) \end{aligned}$$

$$\mathcal{F}_{2N}^{\mathcal{O}}(\theta_1, \dots .\theta_N)_{\mathrm{conn}} = \langle \theta_1, \dots .\theta_N | \mathcal{O}(0,0) | \theta_1, \dots .\theta_N \rangle_{\mathrm{connected}}$$

Pozsgay-Takács (2007): finite volume regularization, to $O\left(\mathrm{e}^{-3mR}\right)$

Pozsgay (2010): proof to all orders

Two-point functions



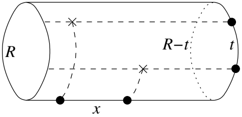
$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T =$$

$$\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n} \langle n\,|\mathcal{O}_1(0,0)|\,m\rangle\, \mathrm{e}^{-(R-t)E_m}\mathrm{e}^{ix(P_m-P_n)} \langle m\,|\mathcal{O}_2(0,0)|\,n\rangle$$

(t: Euclidean time). $|n\rangle$: N particles, $|m\rangle$: M particles

Massive QFT: $E_n \geq N\,m$, $E_m \geq M\,m \Rightarrow$ double series in powers of e^{-mR} and $\mathrm{e}^{-m(R-t)}$

Two-point functions



$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T =$$
$$\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n}\,\langle n\,|\mathcal{O}_1(0,0)|\,m\rangle\,\mathrm{e}^{-(R-t)E_m}\mathrm{e}^{ix(P_m-P_n)}\,\langle m\,|\mathcal{O}_2(0,0)|\,n\rangle$$

(t: Euclidean time). $|n\rangle$: N particles, $|m\rangle$: M particles

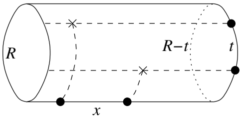
Massive QFT: $E_n \geq N\,m$, $E_m \geq M\,m \Rightarrow$ double series in powers of

$$\mathrm{e}^{-mR} \text{ and } \mathrm{e}^{-m(R-t)}$$

What do we have?

1. Large-distance asymptotics (Altshuler, Konik, Tsvelik 2004)
2. Partial expansion (Essler-Konik 2008,2009 – only works for $N=1$ or $M=1$)

Two-point functions



$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T =$$

$$\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n}\,\langle n\,|\mathcal{O}_1(0,0)|\,m\rangle\,\mathrm{e}^{-(R-t)E_m}\mathrm{e}^{ix(P_m-P_n)}\,\langle m\,|\mathcal{O}_2(0,0)|\,n\rangle$$

(t: Euclidean time). $|n\rangle$: N particles, $|m\rangle$: M particles

Massive QFT: $E_n \geq N\,m$, $E_m \geq M\,m \Rightarrow$ double series in powers of

$$\mathrm{e}^{-mR} \text{ and } \mathrm{e}^{-m(R-t)}$$

What do we have?

1. Large-distance asymptotics (Altshuler, Konik, Tsvelik 2004)
2. Partial expansion (Essler-Konik 2008,2009 – only works for $N = 1$ or $M = 1$)

Goal: to have a systematic expansion!

LeClair-Mussardo proposal (1999)

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\rangle_T =$$
$$\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{\sigma_j=\pm 1}\prod_{j=1}^{N}\left[\int\frac{d\theta_j}{2\pi}f_{\sigma_j}(\theta_j)\mathrm{e}^{-\sigma_j(t\epsilon_j+ixk_j)}\right]$$
$$\times F_N^{\mathcal{O}_1}(\theta_1-i\pi\tilde{\sigma}_1,\dots,\theta_N-i\pi\tilde{\sigma}_N)^{\dagger}F_N^{\mathcal{O}_2}(\theta_1-i\pi\tilde{\sigma}_1,\dots,\theta_N-i\pi\tilde{\sigma}_N)$$

where $\tilde{\sigma}_j=(1-\sigma_j)/2\in\{0,1\}$, $f_{\sigma_j}(\theta_j)=1/(1+e^{-\sigma_j\epsilon(\theta_j)})$,
$\epsilon_j=\epsilon(\theta_j)/R$ és $k_j=k(\theta_j)$

$$\begin{aligned} k(\theta) &= m\sinh\theta+\int d\theta'\delta(\theta-\theta')\rho_1(\theta') \\ 2\pi\rho_1(\theta)(1+\mathrm{e}^{\epsilon(\theta)}) &= m\cosh\theta+\int d\theta'\varphi(\theta-\theta')\rho_1(\theta') \end{aligned}$$

$$F_N^{\mathcal{O}}(\theta_1,\dots,\theta_N)=\langle 0\,|\mathcal{O}(0,0)|\,\theta_1,\dots,\theta_N\rangle$$

LeClair-Mussardo proposal (1999)

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\rangle_T =$$
$$\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{\sigma_j=\pm1}\prod_{j=1}^{N}\left[\int\frac{d\theta_j}{2\pi}f_{\sigma_j}(\theta_j)\mathrm{e}^{-\sigma_j(t\epsilon_j+ixk_j)}\right]$$
$$\times F_N^{\mathcal{O}_1}(\theta_1-i\pi\tilde{\sigma}_1,\ldots,\theta_N-i\pi\tilde{\sigma}_N)^\dagger F_N^{\mathcal{O}_2}(\theta_1-i\pi\tilde{\sigma}_1,\ldots,\theta_N-i\pi\tilde{\sigma}_N)$$

where $\tilde{\sigma}_j=(1-\sigma_j)/2\in\{0,1\}$, $f_{\sigma_j}(\theta_j)=1/(1+e^{-\sigma_j\epsilon(\theta_j)})$,
$\epsilon_j=\epsilon(\theta_j)/R$ és $k_j=k(\theta_j)$

$$\begin{aligned}
k(\theta) &= m\sinh\theta+\int d\theta'\delta(\theta-\theta')\rho_1(\theta') \\
2\pi\rho_1(\theta)(1+\mathrm{e}^{\epsilon(\theta)}) &= m\cosh\theta+\int d\theta'\varphi(\theta-\theta')\rho_1(\theta')
\end{aligned}$$

$$F_N^{\mathcal{O}}(\theta_1,\ldots,\theta_N)=\langle 0\,|\mathcal{O}(0,0)|\,\theta_1,\ldots,\theta_N\rangle$$

Saleur (1999): doubts concerning validity...

Form factors

Asymptotic states:

$$|\theta_1,\dots,\theta_n\rangle = \begin{cases} |\theta_1,\dots,\theta_n\rangle^{in} & : \theta_1 > \theta_2 > \cdots > \theta_n \\ |\theta_1,\dots,\theta_n\rangle^{out} & : \theta_1 < \theta_2 < \cdots < \theta_n \end{cases} \qquad \langle\theta^{'}|\theta\rangle = 2\pi\delta(\theta^{'}-\theta)$$

Factorized scattering:

$$|\theta_1,\dots,\theta_k,\theta_{k+1},\dots,\theta_n\rangle = S(\theta_k-\theta_{k+1})|\theta_1,\dots,\theta_{k+1},\theta_k,\dots,\theta_n\rangle$$

Form factors:

$$F^{\mathcal{O}}_{mn}(\theta^{'}_1,\dots,\theta^{'}_m|\theta_1,\dots,\theta_n)=\langle\theta^{'}_1,\dots,\theta^{'}_m|\mathcal{O}(0,0)|\theta_1,\dots,\theta_n\rangle$$

Form factors

Asymptotic states:

$$|\theta_1,\ldots,\theta_n\rangle = \begin{cases} |\theta_1,\ldots,\theta_n\rangle^{in} & : \theta_1 > \theta_2 > \cdots > \theta_n \\ |\theta_1,\ldots,\theta_n\rangle^{out} & : \theta_1 < \theta_2 < \cdots < \theta_n \end{cases} \qquad \langle\theta^{'}|\theta\rangle = 2\pi\delta(\theta^{'}-\theta)$$

Factorized scattering:

$$|\theta_1,\ldots,\theta_k,\theta_{k+1},\ldots,\theta_n\rangle = S(\theta_k-\theta_{k+1})|\theta_1,\ldots,\theta_{k+1},\theta_k,\ldots,\theta_n\rangle$$

Form factors:

$$F^{\mathcal{O}}_{mn}(\theta^{'}_1,\ldots,\theta^{'}_m|\theta_1,\ldots,\theta_n) = \langle\theta^{'}_1,\ldots,\theta^{'}_m|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_n\rangle$$

Crossing:

$$F^{\mathcal{O}}_{mn}(\theta^{'}_1,\ldots,\theta^{'}_m|\theta_1,\ldots,\theta_n) = F^{\mathcal{O}}_{m-1n+1}(\theta^{'}_1,\ldots,\theta^{'}_{m-1}|\theta^{'}_m+i\pi,\theta_1,\ldots,\theta_n) +$$
$$\sum_{k=1}^{n} 2\pi\delta(\theta^{'}_m-\theta_k)\prod_{l=1}^{k-1} S(\theta_l-\theta_k)F^{\mathcal{O}}_{m-1n-1}(\theta^{'}_1,\ldots,\theta^{'}_{m-1}|\theta_1,\ldots,\theta_{k-1},\theta_{k+1}\ldots,\theta_n)$$

Elementary form factors:

$$F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_n) = \langle 0|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_n\rangle$$

The problem: disconnected pieces

$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T =$$
$$\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n}\langle n|\mathcal{O}_1(0,0)|m\rangle \mathrm{e}^{-(R-t)E_m}\mathrm{e}^{ix(P_m-P_n)}\langle m|\mathcal{O}_2(0,0)|n\rangle$$

The problem: disconnected pieces

$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T =$$
$$\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n} \langle n | \mathcal{O}_1(0,0) | m\rangle \, \mathrm{e}^{-(R-t)E_m} \mathrm{e}^{ix(P_m-P_n)} \langle m | \mathcal{O}_2(0,0) | n\rangle$$

Disconnected pieces: δ and δ^2 terms.

The problem: disconnected pieces

$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T =$$
$$\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n} \langle n | \mathcal{O}_1(0,0) | m\rangle \, \mathrm{e}^{-(R-t)E_m} \mathrm{e}^{ix(P_m-P_n)} \langle m | \mathcal{O}_2(0,0) | n\rangle$$

Disconnected pieces: δ and δ^2 terms.

Well-known treatment: put in box – but here it is important to take into account particle interactions.

The problem: disconnected pieces

$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T =$$
$$\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n} \langle n | \mathcal{O}_1(0,0) | m\rangle \, \mathrm{e}^{-(R-t)E_m} \mathrm{e}^{ix(P_m-P_n)} \langle m | \mathcal{O}_2(0,0) | n\rangle$$

Disconnected pieces: δ and δ^2 terms.

Well-known treatment: put in box – but here it is important to take into account particle interactions.

Why? Because the relevant contribution is subleading in the volume dependence!

The problem: disconnected pieces

$$
\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_T = \\
\frac{1}{Z}\sum_{n,m} \mathrm{e}^{-RE_n} \langle n | \mathcal{O}_1(0,0) | m\rangle \, \mathrm{e}^{-(R-t)E_m} \mathrm{e}^{ix(P_m-P_n)} \langle m | \mathcal{O}_2(0,0) | n\rangle
$$

Disconnected pieces: δ and δ^2 terms.

Well-known treatment: put in box – but here it is important to take into account particle interactions.

Why? Because the relevant contribution is subleading in the volume dependence!
One-point function: all terms are "maximally" disconnected!

$$
\langle \mathcal{O}(t,x)\rangle_T = \frac{1}{Z}\sum_{n} \mathrm{e}^{-RE_n} \langle n | \mathcal{O}(0,0) | n\rangle
$$

Form factors in finite volume I

Spectrum in finite volume:

$$|\{I_1,\ldots,I_n\}\rangle_L \qquad I_1\geq\cdots\geq I_n$$

Bethe-Yang equations:

$$\mathrm{e}^{imL\sinh\tilde{\theta}_k}\prod_{l\neq k}S(\tilde{\theta}_k-\tilde{\theta}_l)=1$$

Using phaseshift $S(\theta)=-\mathrm{e}^{i\delta(\theta)}$:

$$Q_k(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)=mL\sinh\tilde{\theta}_k+\sum_{l\neq k}\delta(\tilde{\theta}_k-\tilde{\theta}_l)=2\pi I_k \quad , \quad k=1,\ldots,n$$

$$E-E_0=\sum_{k=1}^{n}m\cosh\tilde{\theta}_k+O\left(\mathrm{e}^{-\mu L}\right)$$

Density of states:

$$\rho(\theta_1,\ldots,\theta_n)=\det\mathcal{J}^{(n)}\ ,\ \mathcal{J}^{(n)}_{kl}=\frac{\partial Q_k(\theta_1,\ldots,\theta_n)}{\partial\theta_l}$$

Form factors in finite volume II

General case:

$$\langle \{I'_1,\dots,I'_m\}|\mathcal{O}(0,0)|\{I_1,\dots,I_n\}\rangle_L = \frac{F^{\mathcal{O}}_{m+n}(\tilde{\theta}'_m+i\pi,\dots,\tilde{\theta}'_1+i\pi,\tilde{\theta}_1,\dots,\tilde{\theta}_n)}{\sqrt{\rho(\tilde{\theta}_1,\dots,\tilde{\theta}_n)\rho(\tilde{\theta}'_1,\dots,\tilde{\theta}'_m)}} + O(\mathrm{e}^{-\mu L})$$

Form factors in finite volume II

General case:

$$\langle\{I'_1,\dots,I'_m\}|\mathcal{O}(0,0)|\{I_1,\dots,I_n\}\rangle_L = \frac{F^{\mathcal{O}}_{m+n}(\tilde\theta'_m+i\pi,\dots,\tilde\theta'_1+i\pi,\tilde\theta_1,\dots,\tilde\theta_n)}{\sqrt{\rho(\tilde\theta_1,\dots,\tilde\theta_n)\rho(\tilde\theta'_1,\dots,\tilde\theta'_m)}} + O(\mathrm{e}^{-\mu L})$$

Diagonal case: $\{I'_1,\dots,I'_m\}=\{I_1,\dots,I_n\}$

$$\langle\{I_1\dots I_n\}|\mathcal{O}|\{I_1\dots I_n\}\rangle_L = \frac{\sum\limits_{A\subset\{1,2,\dots n\}}\mathcal{F}(A)_L\rho(\{1,\dots,n\}\setminus A)_L}{\rho(\{1,\dots,n\})_L}+O(\mathrm{e}^{-\mu L})$$

where

$$\rho(\{k_1,\dots,k_r\})_L=\rho(\tilde\theta_{k_1},\dots,\tilde\theta_{k_r})$$

and the symmetric evaluation

$$\begin{aligned}\mathcal{F}(\{k_1,\dots,k_r\})_L &= F^s_{2r}(\tilde\theta_{k_1},\dots,\tilde\theta_{k_r})\\ F^s_{2l}(\theta_1,\dots,\theta_l) &= \lim_{\epsilon\to 0} F^{\mathcal{O}}_{2l}(\theta_l+i\pi+\epsilon,\dots,\theta_1+i\pi+\epsilon,\theta_1,\dots,\theta_l)\end{aligned}$$

(Pozsgay-Takács, 2007)

Finite volume as regulator

All matrix elements are finite in finite volume:

$$\delta(\theta - \theta') \to mL \cosh \theta \delta_{II'}$$

Finite volume as regulator

All matrix elements are finite in finite volume:

$$\delta(\theta - \theta') \to mL \cosh\theta \delta_{II'}$$

but: care is needed

$$\begin{aligned} \delta(\theta_1 - \theta_1')\delta(\theta_2 - \theta_2') \quad &\nrightarrow \quad mL\cosh\theta_1 \delta_{I_1 I_1'} mL \cosh\theta_2 \delta_{I_2 I_2'} \\ \text{there are also} \quad &+ \quad O(L) \text{ contributions!} \end{aligned}$$

Finite volume as regulator

All matrix elements are finite in finite volume:

$$\delta(\theta-\theta') \to mL\cosh\theta\,\delta_{II'}$$

but: care is needed

$$\begin{aligned}\delta(\theta_1-\theta_1')\delta(\theta_2-\theta_2') \quad &\not\to \quad mL\cosh\theta_1\delta_{I_1I_1'}\,mL\cosh\theta_2\delta_{I_2I_2'} \\ \text{there are also} \quad &+ \quad O(L) \text{ contributions!}\end{aligned}$$

Solution: take correlator in finite volume

$$\begin{aligned}&\langle\mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle_{T,L} = \\ &\frac{1}{Z(R,L)}\sum_{n,m}\mathrm{e}^{-RE_n(L)}\mathrm{e}^{-(R-t)E_m(L)}\mathrm{e}^{ix(P_m(L)-P_n(L))} \\ &\times\langle n\,|\mathcal{O}_1(0,0)|\,m\rangle_L\,\langle m\,|\mathcal{O}_2(0,0)|\,n\rangle_L \\ &Z(R,L) = \sum_n \mathrm{e}^{-RE_n(L)}\end{aligned}$$

$L\to\infty$ only at the end! (Pozsgay-Takács 2007: 1pt functions)

Reordering the series expansion I

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_{T,L} = \frac{1}{Z}\sum_{N,M} C_{NM}$$

where

$$\begin{aligned} C_{NM} &= \sum_{I_1\dots I_N}\sum_{J_1\dots J_M} \langle \{I_1\dots I_N\}|\mathcal{O}_1(0)|\{J_1\dots J_M\}\rangle_L \times \\ &\quad \langle \{J_1\dots J_M\}|\mathcal{O}_2(0)|\{I_1\dots I_N\}\rangle_L e^{i(P_1-P_2)x} e^{-E_1(R-t)} e^{-E_2 t} \end{aligned}$$

u and v keep track of powers of e^{-mt} and $e^{-m(R-t)}$ (at the end set $u=v=1$)

$$\begin{aligned} \langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_{T,L} &= \frac{1}{Z}\sum_{N,M} u^N v^M C_{NM} \\ Z &= \sum_N (uv)^N Z_N \end{aligned}$$

Reordering the series expansion II

$$Z^{-1} = \sum_N (uv)^N \bar{Z}_N$$

$$\bar{Z}_0 = Z_0 = 1 \qquad \bar{Z}_1 = -Z_1 \qquad \bar{Z}_2 = Z_1^2 - Z_2$$

From this:

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_{T,L} = \sum u^N v^N \tilde{D}_{NM}$$

$$\tilde{D}_{NM} = \sum_l C_{N-l,M-l}\bar{Z}_l$$

Reordering the series expansion II

$$Z^{-1} = \sum_N (uv)^N \bar{Z}_N$$

$$\bar{Z}_0 = Z_0 = 1 \qquad \bar{Z}_1 = -Z_1 \qquad \bar{Z}_2 = Z_1^2 - Z_2$$

From this:

$$\begin{aligned} \langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_{T,L} &= \sum u^N v^N \tilde{D}_{NM} \\ \tilde{D}_{NM} &= \sum_l C_{N-l,M-l}\bar{Z}_l \end{aligned}$$

First few coefficients:

$$\tilde{D}_{1M} = C_{1M} - Z_1 C_{0,M-1}$$
$$\tilde{D}_{2M} = C_{2M} - Z_1 C_{1,M-1} + (Z_1^2 - Z_2) C_{0,M-2}$$

Reordering the series expansion II

$$
\begin{aligned}
Z^{-1} &= \sum_N (uv)^N \bar{Z}_N \\
& \bar{Z}_0 = Z_0 = 1 \qquad \bar{Z}_1 = -Z_1 \qquad \bar{Z}_2 = Z_1^2 - Z_2
\end{aligned}
$$

From this:

$$
\begin{aligned}
\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_{T,L} &= \sum u^N v^N \tilde{D}_{NM} \\
\tilde{D}_{NM} &= \sum_l C_{N-l,M-l} \bar{Z}_l
\end{aligned}
$$

First few coefficients:

$$
\begin{aligned}
\tilde{D}_{1M} &= C_{1M} - Z_1 C_{0,M-1} \\
\tilde{D}_{2M} &= C_{2M} - Z_1 C_{1,M-1} + (Z_1^2 - Z_2) C_{0,M-2}
\end{aligned}
$$

Infinite volume limit:

$$
\begin{aligned}
\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_T &= \lim_{L\to\infty} \langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_{T,L} = \sum_{N,M} D_{NM} \\
D_{NM} &= \lim_{L\to\infty} \tilde{D}_{NM}
\end{aligned}
$$

Partition function

$$Z_1 = \sum_{l} \mathrm{e}^{-mR\cosh\theta} = \int \frac{d\theta}{2\pi} mL\cosh\theta \mathrm{e}^{-mR\cosh\theta} \qquad (\, mL\sinh\theta = 2\pi l \,)$$

Partition function

$$Z_1 = \sum_I \mathrm{e}^{-mR\cosh\theta} = \int \frac{d\theta}{2\pi} mL\cosh\theta \mathrm{e}^{-mR\cosh\theta} \qquad (\, mL\sinh\theta = 2\pi I\,)$$

Two-particle contribution

$$Z_2 = \sum_{I_1<I_2} \mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}$$

$$mL\sinh\theta_1 + \delta(\theta_1-\theta_2) = 2\pi I_1 \quad , \quad mL\sinh\theta_2 + \delta(\theta_2-\theta_1) = 2\pi I_2$$

$$2\sum_{I_1<I_2} = \sum_{I_1,I_2} - \sum_{I_1=I_2} = \int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2}{2\pi} \rho_2(\theta_1,\theta_2) \cdots - \int \frac{d\theta_1}{2\pi} mL\cosh\theta_1 \ldots$$

$$\rho_2(\theta_1,\theta_2) = m^2L^2\cosh\theta_1\cosh\theta_2 + mL(\cosh\theta_1+\cosh\theta_2)\varphi(\theta_1-\theta_2)$$

$$\begin{aligned} Z_2 &= \frac{1}{2}\int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2}{2\pi} \rho_2(\theta_1,\theta_2) \mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)} \\ &\quad -\frac{1}{2}\int \frac{d\theta_1}{2\pi} mL\cosh\theta_1 \mathrm{e}^{-2mR\cosh\theta_1} \end{aligned}$$

Multi-dimensional residue theorem

Performing sums over quantum numbers (at finite L!):
Turn them into integrals!

Multi-dimensional residue theorem

Performing sums over quantum numbers (at finite L!):
Turn them into integrals!

Theorem: *Suppose $g(z)$, $f_1(z), \ldots, f_n(z)$ are analytic functions, where $z = (z_1, \ldots, z_n)$. Let $C \subset \mathbb{C}^n$ be a monomial multi-contour ($C = C_1 \times \cdots \times C_n$). Assume that*

$$f_k(z) = 0 \qquad k = 1, \ldots, n$$

has a solution $z_* = (z_{*1}, \ldots, z_{*n})$ such that for all k z_{*k} is in the interior of C_k and

$$\det\left(\frac{\partial f_k}{\partial z_l}\right)\Bigg|_{z=z_*} \neq 0$$

Then

$$\oint_C \frac{dz_1}{2\pi i} \cdots \frac{dz_n}{2\pi i} \frac{g(z)}{f_1(z) \ldots f_n(z)} = \frac{g(z_*)}{\det\left(\frac{\partial f_k}{\partial z_l}\right)\Big|_{z=z_*}}$$

Multi-dimensional residue theorem

Performing sums over quantum numbers (at finite L!):
Turn them into integrals!
Theorem: *Suppose $g(z)$, $f_1(z), \dots, f_n(z)$ are analytic functions, where $z = (z_1, \dots, z_n)$. Let $C \subset \mathbb{C}^n$ be a monomial multi-contour ($C = C_1 \times \cdots \times C_n$). Assume that*

$$f_k(z) = 0 \qquad k = 1, \dots, n$$

has a solution $z_* = (z_{*1}, \dots, z_{*n})$ such that for all k z_{*k} is in the interior of C_k and

$$\det \left(\frac{\partial f_k}{\partial z_l} \right) \bigg|_{z=z_*} \neq 0$$

Then

$$\oint_C \frac{dz_1}{2\pi i} \dots \frac{dz_n}{2\pi i} \frac{g(z)}{f_1(z) \dots f_n(z)} = \frac{g(z_*)}{\det \left(\frac{\partial f_k}{\partial z_l} \right) \Big|_{z=z_*}}$$

Remarks:

– Standard contour deformations work away from the $\det \partial f = 0$ variety.

– General multi-contour: sum of moniomial ones

Evaluating D_{11} I

$$\begin{aligned}
D_{11} &= \lim_{L\to\infty} \tilde{D}_{11} \\
\tilde{D}_{11} &= C_{11} - Z_1 C_{00} \quad , \quad C_{00} = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \\
C_{11} &= \sum_{I,J} \langle \{I\} | \mathcal{O}_1(0) | \{J\} \rangle_L \langle \{J\} | \mathcal{O}_2(0) | \{I\} \rangle_L \, \mathrm{e}^{i(p_1 - p_2)x} \mathrm{e}^{-E_1(R-t)} \mathrm{e}^{-E_2 t}
\end{aligned}$$

Evaluating D_{11} I

$$
\begin{aligned}
D_{11} &= \lim_{L\to\infty} \tilde{D}_{11} \\
\tilde{D}_{11} &= C_{11} - Z_1 C_{00} \quad , \quad C_{00} = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \\
C_{11} &= \sum_{I,J} \langle \{I\}|\mathcal{O}_1(0)|\{J\}\rangle_L \langle \{J\}|\mathcal{O}_2(0)|\{I\}\rangle_L \; \mathrm{e}^{i(p_1-p_2)x} \mathrm{e}^{-E_1(R-t)} \mathrm{e}^{-E_2 t}
\end{aligned}
$$

where

$$
\begin{aligned}
& mL \sinh\theta = 2\pi I \quad , \quad mL \sinh\theta' = 2\pi J \\
& E_1 = m\cosh\theta,\; p_1 = m\sinh\theta \quad E_2 = m\cosh\theta',\; p_2 = m\sinh\theta' \\
& \langle \{I\}|\mathcal{O}_1(0)|\{J\}\rangle_L = \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta')}{\sqrt{\rho_1(\theta)\rho_1(\theta')}} + \delta_{IJ} \langle \mathcal{O}_1 \rangle \\
& \langle \{J\}|\mathcal{O}_2(0)|\{I\}\rangle_L = \frac{F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta)}{\sqrt{\rho_1(\theta)\rho_1(\theta')}} + \delta_{IJ} \langle \mathcal{O}_2 \rangle
\end{aligned}
$$

Evaluating D_{11} II

From this

$$C_{11} = \sum_{I,J} \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta')F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta)}{\rho_1(\theta)\rho_1(\theta')} \mathrm{e}^{i(p_1 - p_2)x} \mathrm{e}^{-E_1(R-t)} \mathrm{e}^{-E_2 t}$$

$$+ \langle \mathcal{O}_1 \rangle \sum_J \frac{F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta')}{\rho_1(\theta')} \mathrm{e}^{-E_2 R} + \langle \mathcal{O}_2 \rangle \sum_I \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta)}{\rho_1(\theta)} \mathrm{e}^{-E_1 R}$$

$$+ \sum_I \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \mathrm{e}^{-E_1 R}$$

Evaluating D_{11} II

From this

$$C_{11} = \sum_{I,J} \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta') F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta)}{\rho_1(\theta)\rho_1(\theta')} \mathrm{e}^{i(p_1 - p_2)x} \mathrm{e}^{-E_1(R-t)} \mathrm{e}^{-E_2 t}$$

$$+ \langle \mathcal{O}_1 \rangle \sum_J \frac{F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta')}{\rho_1(\theta')} \mathrm{e}^{-E_2 R} + \langle \mathcal{O}_2 \rangle \sum_I \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta)}{\rho_1(\theta)} \mathrm{e}^{-E_1 R}$$

$$+ \sum_I \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \mathrm{e}^{-E_1 R}$$

Last term is $O(L)$, but cancelled by $Z_1 C_{00}$!

$$\tilde{D}_{11} = C_{11} - Z_1 C_{00} \quad , \quad C_{00} = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$$

Evaluating D_{11} II

From this

$$C_{11} = \sum_{I,J} \frac{F_2^{\mathcal{O}_1}(\theta+i\pi,\theta')F_2^{\mathcal{O}_2}(\theta'+i\pi,\theta)}{\rho_1(\theta)\rho_1(\theta')} \mathrm{e}^{i(p_1-p_2)x}\mathrm{e}^{-E_1(R-t)}\mathrm{e}^{-E_2 t}$$
$$+\langle\mathcal{O}_1\rangle \sum_J \frac{F_2^{\mathcal{O}_2}(\theta'+i\pi,\theta')}{\rho_1(\theta')}\mathrm{e}^{-E_2 R} + \langle\mathcal{O}_2\rangle \sum_I \frac{F_2^{\mathcal{O}_1}(\theta+i\pi,\theta)}{\rho_1(\theta)}\mathrm{e}^{-E_1 R}$$
$$+\sum_I \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle\,\mathrm{e}^{-E_1 R}$$

Last term is $O(L)$, but cancelled by $Z_1 C_{00}$!

$$\tilde{D}_{11} = C_{11} - Z_1 C_{00} \quad , \quad C_{00} = \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle$$

End result:

$$\begin{aligned} D_{11} \quad = \quad & \int \frac{d\theta}{2\pi}\int\frac{d\theta'}{2\pi} F_2^{\mathcal{O}_1}(\theta+i\pi,\theta')F_2^{\mathcal{O}_2}(\theta'+i\pi,\theta) \\ & \times \mathrm{e}^{imx(\sinh\theta-\sinh\theta')-m(R-t)\cosh\theta-mt\cosh\theta'} \\ & +(\langle\mathcal{O}_1\rangle F_{2s}^{\mathcal{O}_2}+\langle\mathcal{O}_2\rangle F_{2s}^{\mathcal{O}_1})\int\frac{d\theta}{2\pi}\mathrm{e}^{-mR\cosh\theta}\end{aligned}$$

Evaluating D_{22}

$$\tilde{D}_{22} = C_{22} - Z_1 C_{11} + (Z_1^2 - Z_2) C_{00}$$

Using

$$\tilde{D}_{11} = C_{11} - Z_1 C_{00}$$

gives

$$\tilde{D}_{22} = C_{22} - Z_1 \tilde{D}_{11} - Z_2 C_{00}$$

We need:

$$C_{22} = \sum_{I_1 > I_2} \sum_{J_1 > J_2} \langle \{I_1, I_2\} | \mathcal{O}_1 | \{J_1, J_2\} \rangle_L \langle \{J_1, J_2\} | \mathcal{O}_2 | \{I_1, I_2\} \rangle_L$$
$$\times K_{t,x}^{(R)}(\theta_1, \theta_2; \theta_1', \theta_2')$$
$$K_{t,x}^{(R)}(\theta_1, \theta_2; \theta_1', \theta_2') = \mathrm{e}^{imx(\sinh\theta_1 + \sinh\theta_2 - \sinh\theta_1' - \sinh\theta_2')}$$
$$\times \mathrm{e}^{-m(R-t)(\cosh\theta_1 + \cosh\theta_2)} \mathrm{e}^{-mt(\cosh\theta_1' + \cosh\theta_2')}$$

Separating diagonal pieces:

$$\sum_{I_1 > I_2} \sum_{J_1 > J_2} = \sum_{I_1 > I_2} (\{J_1, J_2\} = \{I_1, I_2\} \text{ tagok}) + \sum_{I_1 > I_2} \sum_{J_1 > J_2}{}'$$

Diagonal pieces I

$$\langle\{I_1,I_2\}|\mathcal{O}|\{I_1,I_2\}\rangle_L = \frac{F_{4s}^{\mathcal{O}}(\theta_1,\theta_2)+\rho_1(\theta_1)F_{2c}^{\mathcal{O}}+\rho_1(\theta_2)F_{2c}^{\mathcal{O}}+\rho_2(\theta_1,\theta_2)\langle\mathcal{O}\rangle}{\rho_2(\theta_1,\theta_2)}$$

Put into diagonal part:

$$\sum_{I_1>I_2}\langle\{I_1,I_2\}|\mathcal{O}_1|\{I_1,I_2\}\rangle_L\langle\{I_1,I_2\}|\mathcal{O}_2|\{I_1,I_2\}\rangle_L\,\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}$$

$F_{4s}F_{4s}$ terms:

$$\sum_{I_1>I_2}\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)^2}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)=$$

$$\left(\frac{1}{2}\sum_{I_1,I_2}-\sum_{I_1=I_2}\right)\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)^2}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)=$$

$$\frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)$$

$$-\int\frac{d\theta_1}{2\pi}\frac{\mathrm{e}^{-2mR(\cosh\theta_1)}mL\cosh\theta_1}{\rho_2(\theta_1,\theta_1)^2}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_1)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_1)$$

$O(L^{-2})+O(L^{-3})$: 0 for $L\to\infty$.

Diagonal pieces II

Similarly for other pieces, the total is:

$$C_{22}^{\mathrm{diag}} = \int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2}{2\pi} \Bigg[\frac{1}{2} \mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)} F_{2c}^{\mathcal{O}_1} F_{2c}^{\mathcal{O}_2} \frac{(\cosh\theta_1+\cosh\theta_2)^2}{\cosh\theta_1\cosh\theta_2}$$

$$+\frac{1}{2}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\left(m^2L^2\cosh\theta_1\cosh\theta_2 + mL(\cosh\theta_1+\cosh\theta_2)\varphi(\theta_1-\theta_2)\right)$$

$$\times \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle + mL\cosh\theta_1 \mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\left(\langle\mathcal{O}_1\rangle F_{2c}^{\mathcal{O}_2} + \langle\mathcal{O}_2\rangle F_{2c}^{\mathcal{O}_1}\right)\Bigg]$$

$$-\frac{1}{2}\int\frac{d\theta}{2\pi}\mathrm{e}^{-2mR\cosh\theta}\left[mL\cosh\theta\langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle + 2\left(\langle\mathcal{O}_1\rangle F_{2c}^{\mathcal{O}_2}+\langle\mathcal{O}_2\rangle F_{2c}^{\mathcal{O}_1}\right)\right]$$

"Counter terms":

$$-Z_2C_{00} = \frac{1}{2}\int\frac{d\theta}{2\pi} mL\cosh\theta \mathrm{e}^{-2mR\cosh\theta}\langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle$$

$$-\frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\big[m^2L^2\cosh\theta_1\cosh\theta_2$$

$$+mL(\cosh\theta_1+\cosh\theta_2)\varphi(\theta_1-\theta_2)\big]\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle$$

$$-Z_1D_{11} = -Z_1\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}F_2^{\mathcal{O}_1}(\theta_1+i\pi,\theta_2)F_2^{\mathcal{O}_2}(\theta_1,\theta_2+i\pi)$$

$$\times\mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2)}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2}$$

$$-Z_1\int\frac{d\theta}{2\pi}\left(F_{2c}^{\mathcal{O}_1}\langle\mathcal{O}_2\rangle + F_{2c}^{\mathcal{O}_2}\langle\mathcal{O}_1\rangle\right)\mathrm{e}^{-mR\cosh\theta}$$

Diagonal part III

Divergent parts drop out:

$$
\begin{aligned}
\mathrm{D}_{22}^{(\mathrm{diag})} &= \frac{1}{2}\int \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\Big(F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)\left\langle\mathcal{O}_2\right\rangle \\
&\quad +F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)\left\langle\mathcal{O}_1\right\rangle\Big) \\
&+ \frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}F_{2c}^{\mathcal{O}_1}F_{2c}^{\mathcal{O}_2}\frac{(\cosh\theta_1+\cosh\theta_2)^2}{\cosh\theta_1\cosh\theta_2} \\
&- \int\frac{d\theta}{2\pi}\mathrm{e}^{-2mR\cosh\theta}\left(\left\langle\mathcal{O}_1\right\rangle F_{2c}^{\mathcal{O}_2}+\left\langle\mathcal{O}_2\right\rangle F_{2c}^{\mathcal{O}_1}\right)
\end{aligned}
$$

But: $-Z_1 D_{11}$ remains!

$$
\begin{aligned}
-Z_1\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}&F_2^{\mathcal{O}_1}(\theta_1+i\pi,\theta_2)F_2^{\mathcal{O}_2}(\theta_1,\theta_2+i\pi) \\
&\times\mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2)}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2}
\end{aligned}
$$

and will regulate the nondiagonal part.

Nondiagonal part of D_{22} I

$$C_{22}^{(\mathrm{nondiag})} = \sum_{I_1>I_2} \sum_{J_1>J_2}{}' \langle\{I_1,I_2\}|\mathcal{O}_1|\{J_1,J_2\}\rangle_L \langle\{J_1,J_2\}|\mathcal{O}_2|\{I_1,I_2\}\rangle_L \times K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2')$$

$$K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2') = \mathrm{e}^{imx(\sinh\theta_1+\sinh\theta_2-\sinh\theta_1'-\sinh\theta_2')}\mathrm{e}^{-m(R-t)(\cosh\theta_1+\cosh\theta_2)} \times \mathrm{e}^{-mt(\cosh\theta_1'+\cosh\theta_2')}$$

$$\langle\{I_1,I_2\}|\mathcal{O}_1|\{J_1,J_2\}\rangle_L = \frac{F_4^{\mathcal{O}_1}(\theta_2+i\pi,\theta_1+i\pi,\theta_1',\theta_2')}{\rho_2(\theta_1,\theta_2)^{1/2}\rho_2(\theta_1',\theta_2')^{1/2}}$$

$$\langle\{J_1,J_2\}|\mathcal{O}_2|\{I_1,I_2\}\rangle_L = \frac{F_4^{\mathcal{O}_2}(\theta_2'+i\pi,\theta_1'+i\pi,\theta_1,\theta_2)}{\rho_2(\theta_1,\theta_2)^{1/2}\rho_2(\theta_1',\theta_2')^{1/2}}$$

$$\begin{aligned}
Q_1(\theta_1,\theta_2) &= mL\sinh\theta_1+\delta(\theta_1-\theta_2)=2\pi I_1\\
Q_2(\theta_1,\theta_2) &= mL\sinh\theta_2+\delta(\theta_2-\theta_1)=2\pi I_2\\
Q_{1'}(\theta_1',\theta_2') &= mL\sinh\theta_1'+\delta(\theta_1'-\theta_2')=2\pi J_1\\
Q_{2'}(\theta_1',\theta_2') &= mL\sinh\theta_2'+\delta(\theta_2'-\theta_1')=2\pi J_2
\end{aligned}$$

Nondiagonal part of D_{22} II

Using the residue theorem

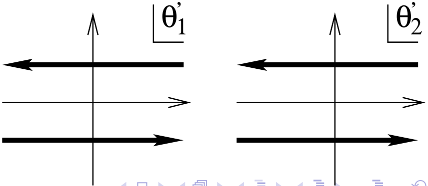
$$\begin{aligned}
&\langle \{I_1, I_2\} |\mathcal{O}_1| \{J_1, J_2\}\rangle_L \langle \{J_1, J_2\} |\mathcal{O}_2| \{I_1, I_2\}\rangle_L K_{t,x}^{(R)}(\theta_1, \theta_2; \theta_1', \theta_2') \\
&= \frac{1}{\rho_2(\theta_1, \theta_2)} \oint \oint_{C_{J_1 J_2}} \frac{d\theta_1'}{2\pi} \frac{d\theta_2'}{2\pi} F_4^{\mathcal{O}_1}(\theta_2 + i\pi, \theta_1 + i\pi, \theta_1', \theta_2') \\
&\times F_4^{\mathcal{O}_2}(\theta_2' + i\pi, \theta_1' + i\pi, \theta_1, \theta_2) \frac{K_{t,x}^{(R)}(\theta_1, \theta_2; \theta_1', \theta_2')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1', \theta_2')} + 1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1', \theta_2')} + 1\right)}
\end{aligned}$$

where $C_{J_1 J_2}$ surrounds the solution of

$$\begin{aligned}
Q_{1'}(\theta_1', \theta_2') &= mL \sinh \theta_1' + \delta(\theta_1' - \theta_2') = 2\pi J_1 \\
Q_{2'}(\theta_1', \theta_2') &= mL \sinh \theta_2' + \delta(\theta_2' - \theta_1') = 2\pi J_2
\end{aligned}$$

This is the contour we want to "open":

$$\frac{1}{2} \sum_{J_1 \neq J_2} C_{J_1 J_2} \to C_1 \times C_2$$



Nondiagonal part of D_{22} III

But: new singularities encountered.

1. QF: partly from Q, partly from F

$$\begin{aligned}
\theta_1' &= \theta_1 &\quad \text{and} \quad& Q_{2'}(\theta_1', \theta_2') = 2\pi J_2 \\
\theta_1' &= \theta_2 &\quad \text{and} \quad& Q_{2'}(\theta_1', \theta_2') = 2\pi J_2 \\
\theta_2' &= \theta_1 &\quad \text{and} \quad& Q_{1'}(\theta_1', \theta_2') = 2\pi J_1 \\
\theta_2' &= \theta_2 &\quad \text{and} \quad& Q_{1'}(\theta_1', \theta_2') = 2\pi J_1
\end{aligned}$$

2. FF: from F

$$\begin{aligned}
\theta_1' = \theta_2' &= \theta_1 \\
\theta_1' = \theta_2' &= \theta_2
\end{aligned}$$

3. QQ (vanishes for $L \to \infty$):

$$\begin{aligned}
\theta_1' &= \theta_1 &\quad \text{és} \quad& \theta_2' = \theta_2 \\
\theta_1' &= \theta_2 &\quad \text{és} \quad& \theta_2' = \theta_1
\end{aligned}$$

Nondiagonal part of D_{22} IV

So:

$$\sum_{J_1>J_2}{}' \oint\oint_{C_{J_1J_2}} \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} = \frac{1}{2}\oint\oint_{C} \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} - (FF \text{ terms}) - (QF \text{ terms})$$

where $C = C_1 \times C_2$ (the open multi-contour).
($J_1 = J_2$ is not a singularity, because form factors vanish).

Kinematical singularity axiom

$$-i \operatorname*{Res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(\theta+i\pi, \theta', \theta_1, \ldots, \theta_n) = \left(1 - \prod_{k=1}^{n} S(\theta' - \theta_k)\right) F^{\mathcal{O}}_{n}(\theta_1, \ldots, \theta_n)$$

can be used to compute QF and QQ residues. (Tedious and the result is lengthy, but straightforward).

Nondiagonal part of D_{22} V

Divergent part comes from second order poles in QF terms:

$$
\begin{aligned}
\mathrm{D}_{QF} \quad = \quad & \iint \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\int \frac{d\theta_2'}{2\pi} F_2^{O_1}(\theta_2+i\pi,\theta_2')F_2^{O_2}(\theta_2'+i\pi,\theta_2) \\
\times \quad & \mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_2')}\mathrm{e}^{-mR\cosh\theta_1}\mathrm{e}^{-m(R-t)\cosh\theta_2}\mathrm{e}^{-mt\cosh\theta_2'} \\
\times \quad & \Big((mx\cosh\theta_1 - imt\sinh\theta_1)(1-S(\theta_2'-\theta_1)S(\theta_1-\theta_2)) \\
& +\varphi(\theta_1-\theta_2')S(\theta_2'-\theta_1)S(\theta_1-\theta_2) + mL\cosh\theta_1\Big) \\
- \quad & \int \frac{d\theta_1}{2\pi}\int \frac{d\theta_2'}{2\pi} F_2^{O_1}(\theta_1+i\pi,\theta_2')F_2^{O_2}(\theta_2'+i\pi,\theta_1) \\
\times \quad & \mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2')}\mathrm{e}^{-m(2R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2'}
\end{aligned}
$$

and is exactly cancelled by the remaining „counter-term”:

$$
-\int \frac{d\theta}{2\pi} mL\cosh\theta \mathrm{e}^{-mR\cosh\theta}\int \frac{d\theta_1}{2\pi}\int \frac{d\theta_2}{2\pi} F_2^{\mathcal{O}_1}(\theta_1+i\pi,\theta_2)F_2^{\mathcal{O}_2}(\theta_1,\theta_2+i\pi)
$$
$$
\times \mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2)}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2}
$$

Conclusion

1. Systematic form factor expansion for thermal correlators
   Consistency: $L \to \infty$ limit is finite
   Explicitly constructed: D_{0N}, D_{M0}, D_{1N}, D_{M1} and D_{22}.
2. Comparison to LeClair-Mussardo 2pt function formula:
   2.1 LM expression not even well-defined
   2.2 Higher corrections look different
3. Method was also applied to
   3.1 $T = 0$ 3-point function
   3.2 second order form factor perturbation theory
4. Kormos-Pozsgay (2010): residue method works for one-point functions of bulk fields on an interval with boundaries.

Open issues

1. Is it possible to sum up the T-dependence in "dressing factors" (a la LeClair–Mussardo)
   1-point function: OK (but we knew that beforehand)
   2-point function: need higher terms (e.g. D_{2N}) to verify: in progress...
2. Can the method be simplified?
   (Lots of residue-crunching and redundant terms)
3. Applications: need extension of finite volume form factors to nondiagonal theories.
   Work in progress... result expected to be under the Christmas tree!

   E.g.: $O(3)$ σ-model (Essler-Konik: $M = 1,\ N = 2$ term for spin-spin correlator).