Pohlmeyer reduced theory for $AdS_5 \times S^5$ superstring

Arkady Tseytlin

B. Hoare and A.T., arXiv:1104.2423

Y. Iwashita, R. Roiban and A.T., to appear

also talk of Luis Miramontes

Aim :

solve string theory in $AdS_5 \times S^5$ from first principles
– conformal invariance, supersymmetry
and integrability

(i) find S-matrix and justify
Asymptotic Bethe Ansatz for the spectrum

(ii) understand theory on cylinder (closed string): TBA

Quantum string theory in $AdS_5 \times S^5$:

GS superstring on $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$
analogy with exact solution of $O(n)$ model (Zamolodchikovs)
or principal chiral model (Polyakov-Wiegmann) ?
2d CFT – no mass generation (mass scale from gauge fixing)

problem of direct approach:
lack of manifest 2d Lorentz symmetry :
S-matrix depends on two rapidities (not on difference)
symmetry constraints on it are not manifest, ...

An alternative approach?
Classically equivalent 2d Lorentz invariant action
describing same physical degrees of freedom

"Pohlmeyer reduction" :

reformulation of gauge-fixed $AdS_5 \times S^5$ superstring
in terms of current-type variables
preserving 2d Lorentz invariance

classically equivalent (equivalent integrable structure);
what about relation at the quantum level?

a way towards exact solution of quantum $AdS_5 \times S^5$ superstring?

Pohlmeyer-reduced theory:

Integrable + 2d scale-invariant (UV finite) model
a fermionic generalization of non-abelian Toda theory

• intimately related to (classical) $AdS_5 \times S^5$ GS model
• action quadratic in fermions with standard 2d kinetic terms
(hidden 2d susy, ...)
• has 2d Lorentz invariant S-matrix
for an equivalent set of 8+8 physical massive excitations:
an alternative interacting generalization of same free theory
• very special UV finite massive integrable model:
deserves study regardless question about equivalence
to $AdS_5 \times S^5$ superstring at quantum level:
find its exact solution?

Some history

K. Pohlmeyer (1976):
Discovery of integrability (existence of ∞ of conservation laws)
of *classical* O(3) sigma model via relation to sine-Gordon theory;
O(4) sigma model$\longrightarrow$ complex sine-Gordon theory.
Integrability of O(n) model: Backlund transformations
to generate solutions and higher conserved charges.

But why reduction relevant?
Assumed classical 2d conf. inv. which is broken at quantum level

Quantum O(3) and sin-Gordon theories are different
but integrability itself extends to quantum level
[Polyakov (1977); Zamolodchikov, Zamolodchikov (1979)]

Pohlmeyer reduction was not used much in the next 20 years...
but came to light in the context of string theory :

Technical tool : classical solutions

• construction of classical string solutions in
constant-curvature spaces – de Sitter and anti de Sitter
[Barbashov, Nesterenko, 1981; de Vega, Sanchez, 1993]

• construction of classical string solutions in $AdS_5 \times S^5$
representing semiclassical string states
[Hofman, Maldacena,2006; Dorey et al,2006; Jevicki et al,2007;
Hoare, Iwashita, A.T., 2009; Hollowood, Miramontes, 2009; ...]

• construction of euclidean open-string world-surfaces related
to Wilson loops (SYM scattering amplitudes at strong coupling)
[Alday, Maldacena, 2009; Alday, Gaiotto, Maldacena, 2009;
Dorn et al, 2009; Jevicki, Jin, 2009, ...]

More fundamental role : relation to quantum string theory?
Pohlmeyer reduction of $AdS_5 \times S^5$ string:
reformulation in terms of integrable massive theory
[Grigoriev, A.T, 2007; Mikhailov, Schafer-Nameki, 2007]

string sigma model is UV finite:
PR may lead to an equivalent theory also at quantum level?
a way to exact solution of $AdS_5 \times S^5$ superstring?

- UV finiteness of reduced theory [Roiban, A.T., 2009]
- equivalence of 1-loop quantum partition functions of string and reduced theory [Hoare, Iwashita, A.T., 2009]
- perturbative S-matrix of reduced theory: similarity to $AdS_5 \times S^5$ magnon S-matrix; q-deformed susy [Hoare and A.T., 2009-2011]
- comparison of soliton spectra and soliton S-matrices [Hollowood and Miramontes, 2010, 2011; Hoare et al, 2011]
- hidden 2d susy [Grigoriev, A.T.; Schmidtt; Hollowood, Miramontes, 2011; Goykhman, Ivanov, 2011]

Pohlmeyer reduction

Original example: S^2-sigma model $\longrightarrow$ Sine-Gordon theory

$$L = \partial_+ X^m \partial_- X^m - \Lambda (X^m X^m - 1)\,, \qquad m = 1, 2, 3$$

Equations of motion:

$$\partial_+ \partial_- X^m + \Lambda X^m = 0\,, \quad \Lambda = \partial_+ X^m \partial_- X^m\,, \quad X^m X^m = 1$$

Stress tensor: $\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$

$$\mathrm{T}_{+-} = 0\,, \quad \partial_+ \mathrm{T}_{--} = 0\,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies $\mathrm{T}_{++} = f(\sigma_+)$, $\mathrm{T}_{--} = h(\sigma_-)$
using the conformal transformations $\sigma_\pm \longrightarrow F_\pm(\sigma_\pm)$ can set

$$\partial_+ X^m \partial_+ X^m = \mu^2\,, \qquad \partial_- X^m \partial_- X^m = \mu^2\,, \qquad \mu = \mathrm{const}$$

3 unit vectors in 3-dimensional Euclidean space:

$$X^m\,, \qquad X^m_+ = \mu^{-1} \partial_+ X^m\,, \qquad X^m_- = \mu^{-1} \partial_- X^m$$

X^m is orthogonal to X_+^m and X_-^m $(X^m \partial_\pm X^m = 0)$
remaining $SO(3)$ invariant quantity is scalar product

$$\partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi$$

then

$$\partial_+ \partial_- \varphi + \frac{\mu^2}{2} \sin 2\varphi = 0$$

following from sine-Gordon action [Pohlmeyer, 1976]

$$\widetilde{L} = \partial_+ \varphi \partial_- \varphi + \frac{\mu^2}{2} \cos 2\varphi$$

2d Lorentz invariant despite explicit constraints

Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating folded string on S^2:
"giant magnon" in the $J = \infty$ limit [Hofman, Maldacena 06]

Analogous construction for S^3 model gives
Complex sine-Gordon model (Pohlmeyer; Lund, Regge 76)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

φ, θ are $SO(4)$-invariants:

$$\mu^2\cos 2\varphi = \partial_+X^m\partial_-X^m$$
$$\mu^3\sin^2\varphi\,\partial_\pm\theta = \mp\tfrac{1}{2}\epsilon_{mnkl}X^m\partial_+X^n\partial_-X^k\partial_\pm^2X^l$$

In the case of AdS_2 or AdS_3:
replace $\sin\varphi \to \sinh\phi$, ...

String-theory interpretation: string on $R_t \times S^n$

(i) conformal gauge
(ii) $t = \mu\tau$ to fix conformal diff's:

$\partial_\pm X^m \partial_\pm X^m = \mu^2$ are Virasoro constraints
e.g., reduced theory for string on $R_t \times S^3$

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

Similar construction for AdS_n case:
string on $AdS_n \times S^1_\psi$ with $\psi = \mu\tau$
e.g., reduced theory for string on $AdS_3 \times S^1$

$$\widetilde{L} = \partial_+\phi\partial_-\phi + \coth^2\varphi\,\partial_+\chi\partial_-\chi - \frac{\mu^2}{2}\cosh 2\phi$$

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to original string coordinates
- Reduced and string theories: equivalent as classical integrable systems: Lax pairs are gauge-equivalent
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- Reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables: "blind" to original global symmetry
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge with 2d Lorentz symmetry unbroken

PR for bosonic string on $R_t \times F/G$

F/G-coset sigma model: symmetric space

$$\mathrm{f} = \mathrm{p} \oplus \mathrm{g} \,, \qquad [\mathrm{g}, \mathrm{g}] \subset \mathrm{g} \,, \qquad [\mathrm{g}, \mathrm{p}] \subset \mathrm{p} \,, \qquad [\mathrm{p}, \mathrm{p}] \subset \mathrm{g}$$

$$J = f^{-1} df = \mathcal{A} + P \,, \qquad \mathcal{A} \in \mathrm{g} \,, \qquad P \in \mathrm{p}$$

$$L(f) = -\mathrm{Tr}(P_+ P_-) \,, \qquad f \in F$$

G gauge transformations: $f \to fg, \quad g \in G$
global F symmetry: $f \to uf, \quad u \in F$
classical conformal invariance

Currents $J = \mathcal{A} + P$ as fundamental variables:
EOM : $D_+ P_- = 0 \,, \quad D_- P_+ = 0 \,, \qquad D = d + [\mathcal{A}, \ ]$
Maurer-Cartan : $D_- P_+ - D_+ P_- + [P_+, P_-] + \mathcal{F}_{+-} = 0$
Virasoro : $\mathrm{Tr}(P_+ P_+) = -\mu^2 \,, \quad \mathrm{Tr}(P_- P_-) = -\mu^2$

Main idea: first solve Virasoro and EOM;
then find reduced action giving eqs. resulting from MC

gauge fixing that solves $\mathrm{Tr}(P_+P_+) = -\mu^2$

$$P_+ = \mu\, T = \mathrm{const}\ , \qquad T \in \mathrm{p} = \mathrm{f} \ominus \mathrm{g}, \qquad \mathrm{Tr}(\,TT) = -1$$

- choice of special element T : decomposition of f

$$\mathrm{f} = \mathrm{p} \oplus \mathrm{g}\,, \qquad \mathrm{p} = \,T \oplus \mathrm{n}\,, \qquad \mathrm{g} = \mathrm{m} \oplus \mathrm{h}\,, \qquad [\,T, \mathrm{h}] = 0$$

- T determines h , i.e. defines subgroup $H \subset G$
- $\mathrm{Tr}(P_-P_-) = -\mu^2$ is solved by introducing $g \in G$

$$P_- = \mu\, g^{-1}Tg$$

$D_-P_+ = 0$ is solved by $\mathcal{A}_- = (\mathcal{A}_-)_{\mathrm{h}} \equiv A_-$
$D_+P_- = 0$ is solved by $\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1}A_+ g$

- thus new "current" variables:

$$g \in G\ , \qquad A_+, A_- \in \mathrm{h}\,, \qquad [T, A_\pm] = 0$$

Remarkably, remaining MC eqs on $g,\ A_\pm$ follow from
G/H gauged WZW action with integrable potential:

$$L = -\frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term}$$
$$-\mathrm{Tr}\Big(A_+\,\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\Big)$$
$$-\mu^2\mathrm{Tr}(T g^{-1} T g)$$

Pohlmeyer-reduced theory for F/G coset sigma model
[Bakas, Park, Shin 95; Grigoriev, A.T. 07; Miramontes 08]

equivalent eqs of motion; equivalent integrable structure

special case of non-abelian Toda theory:
"symmetric space Sine-Gordon model"
[Hollowood, Miramontes et al 96]

potential term: equal to original coset sigma model action

Comments :

- Reduced equations of motion in "on-shell" gauge $A_\pm = 0$:

$$\partial_-(g^{-1}\partial_+ g) - \mu^2 [T, g^{-1}Tg] = 0$$

$$(g^{-1}\partial_+ g)_{\rm h} = 0 \ , \qquad\qquad (\partial_- g g^{-1})_{\rm h} = 0$$

- $F/G = S^n = \frac{SO(n+1)}{SO(n)}, \ \ G/H = \frac{SO(n)}{SO(n-1)}$:

$H = SO(n-1)$ gauge fixing on g (Euler angles)
and integrating out $A_\pm$: generalized SG model

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

metric G_{pq} in kinetic term for $n = 2, 3, 4, ...$

$$ds_2^2 = d\varphi^2 \ , \qquad\qquad ds_3^2 = d\varphi^2 + \cot^2\varphi \, d\theta^2$$

$$ds_4^2 = d\varphi^2 + \cot^2\varphi \, (d\theta_1 + \cot\theta_1 \tan\theta_2 d\theta_2)^2 + \tan^2\varphi \, \frac{d\theta_2^2}{\sin^2\theta_1}$$

String Theory in $AdS_5 \times S^5$

$$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)} = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$$

GS superstring:

replace $\frac{\widehat{F}}{G} = \frac{\mathrm{SuperPoincare}}{\mathrm{Lorentz}}$ in flat case by

$$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)}$$

basic superalgebra $\widehat{\mathrm{f}} = psu(2,2|4)$

bosonic part $\mathrm{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

$$\widehat{\mathrm{f}} = \mathrm{f}_0 \oplus \mathrm{f}_1 \oplus \mathrm{f}_2 \oplus \mathrm{f}_3 \,, \qquad\qquad [\mathrm{f}_i, \mathrm{f}_j] \subset \mathrm{f}_{i+j \bmod 4}$$

$$\mathrm{f}_0 = \mathrm{g} = sp(2,2) \oplus sp(4), \qquad \mathrm{f}_2 = AdS_5 \times S^5$$

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a} \,, \qquad f \in \widehat{F}$$

$$\mathcal{A} \in \mathrm{f}_0, \quad Q_1 \in \mathrm{f}_1, \quad P \in \mathrm{f}_2, \quad Q_2 \in \mathrm{f}_3$$

GS action: $I_{GS} = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \, L_{\rm GS}$

$$L_{\rm GS} = {\rm STr}(\sqrt{-g} g^{ab} P_a P_b + \varepsilon^{ab} Q_{1a} Q_{2b})$$

conformal gauge : $\sqrt{-g} g^{ab} = \eta^{ab}$

$$L_{\rm GS} = {\rm STr}[P_+ P_- + \frac{1}{2}(Q_{1+} Q_{2-} - Q_{1-} Q_{2+})]$$

Virasoro : ${\rm STr}(P_+ P_+) = 0 \,, \qquad {\rm STr}(P_- P_-) = 0$

EOM :
$$\partial_+ P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0$$
$$\partial_- P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0$$
$$[P_+, Q_{1-}] = 0 \,, \qquad [P_-, Q_{2+}] = 0$$

MC : $\partial_- J_+ - \partial_+ J_- + [J_-, J_+] = 0$

now apply Pohlmeyer reduction

Pohlmeyer reduced theory

Bosons :
Virasoro solved by fixing special G-gauge
and residual conformal diffs

$$P_+ = \mu T , \qquad P_- = \mu g^{-1} T g , \qquad \mu = \text{const}$$

$$g \in G = Sp(2,2) \times Sp(4)$$

• μ= an arbitary scale parameter – remnant of fixing residual conformal diffeomorphisms (cf. p^+ in l.c. gauge)

• T – fixed constant matrix = diag$(I, -I, I, -I)$, $\text{Str}\, T^2 = 0$
• selects $H \in G$: $[T, h] = 0, \ h \in H$
$H = SU(2) \times SU(2) \times SU(2) \times SU(2)$
• residual H gauge invariance of e.o.m. for $g, A_\pm$

• new bosonic variables :
$g \in G = Sp(2,2) \times Sp(4)$
$A_\pm \in \mathrm{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$

Fermions :

impose partial κ-symmetry gauge

$$Q_{1-} = 0 \,, \qquad Q_{2+} = 0$$
$$\Psi_1 \equiv Q_{1+} \in \mathfrak{f}_1 \,, \qquad \Psi_2 \equiv g Q_{2-} g^{-1} \in \mathfrak{f}_3$$

residual κ-symmetry fixed by demanding $\{\Psi_{1,2}, T\} = 0$

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\perp} + \widehat{\mathfrak{f}}^{\|} \,, \qquad [\widehat{\mathfrak{f}}^{\perp}, \ T] = 0 \,, \qquad \{\widehat{\mathfrak{f}}^{\|}, \ T\} = 0$$

new fermionic variables :

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}} \Psi_1^{\|} \,, \qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}} \Psi_2^{\|}$$

$\Psi_{_{R,L}}$ expressed in terms of real Grassmann
2×2 matrices $\xi_{_{R,L}}$ and $\eta_{_{R,L}}$: 8+8=16 components

Remarkably, exists local action for $g, A_\pm, \Psi_{R,L}$
reproducing remaining classical equations:

Gauged WZW model for

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

with integrable potential and fermionic terms:

$$\widetilde{L} = L_{\mathrm{gWZW}}(g,A) + \mu^2\, \mathrm{Str}(g^{-1}TgT)$$

$$+\, \mathrm{Str}\left(\Psi_L T D_+ \Psi_L + \Psi_R T D_- \Psi_R + \mu\, g^{-1}\Psi_L g \Psi_R\right)$$

- fields $g, A_\pm, \Psi_{R,L}$ are 8×8 supermatrices, e.g.
$g = \mathrm{diag}(a,b)\ ,\qquad a \in Sp(2,2),\quad b \in Sp(4)$
- $T = \frac{i}{2}\mathrm{diag}(1,1,-1,-1,1,1,-1,-1);$
$[T,h]=0,\ \ h\in H = [SU(2)]^4$
- $D_\pm \Psi = \partial_\pm \Psi + [A_\pm, \Psi],\qquad A_\pm \in \mathrm{h}$
invariance under H gauge transformations
$g' = h^{-1}gh,\quad A'_\pm = h^{-1}(A_\pm + \partial_\pm)h,\quad \Psi'_{L,R} = h^{-1}\Psi_{L,R}h$

Comments:

- integrable model classically equivalent to GS string
- 2d Lorentz invariant action with $\Psi_{_R}, \Psi_{_L}$ as 2d Majorana spinors with standard kinetic terms; action quadratic in fermions (cf. GS string)
- 8 real bosonic and 16 real fermionic independent variables; fermions link bosons from $Sp(2,2) \times Sp(4)$
- 2d supersymmetry: in $AdS_n \times S^n$ with $n=2$ (equivalent to $N=2$ super sine-Gordon); non-local in $n=3,5$ cases
- μ-dependent interaction terms are equal to GS Lagrangian; gWZW terms are to produce MC eqs. (path integral derivation?)
- linearisation of e.o.m. in the gauge $A_\pm = 0$ around $g = \mathbf{1}$: gives 8+8 bosonic and fermionic d.o.f. with mass μ same as in string l.c. gauge action with $\mu \sim J$ (BMN limit)
- Action $I_{\rm PR} = \frac{k}{8\pi} \int d^2\sigma\, \widetilde{L}$: meaning of k?

Equations of motion in $A_\pm = 0$ gauge:
fermionic generalization of non-abelian Toda equations

$$
\begin{aligned}
&\partial_-(g^{-1}\partial_+ g) + \mu^2[g^{-1}Tg, T] + \mu[g^{-1}\Psi_{_L} g, \Psi_{_R}] = 0\\
&T\partial_-\Psi_{_R} + \tfrac{1}{2}\mu(g^{-1}\Psi_{_L} g)^{\|} = 0\\
&T\partial_+\Psi_{_L} + \tfrac{1}{2}\mu(g\Psi_{_R} g^{-1})^{\|} = 0\\
&(g^{-1}\partial_+ g - \tfrac{1}{2}[[T,\Psi_{_R}],\Psi_{_R}])_{\rm h} = 0\\
&(g\partial_- g^{-1} - \tfrac{1}{2}[[T,\Psi_{_L}],\Psi_{_L}])_{\rm h} = 0
\end{aligned}
$$

PR model:
resembles both WZW model based on a supergroup
and 2d supersymmetric WZW model
(fermions have standard 1-st order kinetic terms)

2d supersymmetry?
GS: target space susy + kappa-symmetry
l.c. gauge in flat space: fermions as 2d scalars $\rightarrow$ 2d spinors

Similar lower-dimensional models

$AdS_2 \times S^2$:

$$\frac{\widehat{F}}{G} = \frac{PSU(1,1|2)}{SO(1,1) \times SO(2)}$$

$G = SO(1,1) \times SO(2), \qquad H$=trivial

PR: [sin-Gordon + sinh-Gordon]   + fermions

$AdS_3 \times S^3$:

$$\frac{\widehat{F}}{G} = \frac{PS\big[U(1,1|2) \times U(1,1|2)\big]}{U(1,1) \times U(2)}$$

$G = U(1,1) \times U(2), \qquad H = [U(1)]^4$

PR: [complex sin-Gordon + complex sinh-Gordon]   + fermions

PR model for superstring on $AdS_2 \times S^2$

PR Lagrangian: same as $n = 2$ supersymmetric sine-Gordon

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$
$$+ \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho$$
$$- 2\mu\left[\cosh\phi\ \cos\varphi\ (\beta\nu + \gamma\rho) + \sinh\phi\ \sin\varphi\ (\beta\rho - \gamma\nu)\right]$$

equivalent to:

$$L = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2 + \psi^*_L\partial_+\psi_L + \psi^*_R\partial_-\psi_R$$
$$+\left[W''(\Phi)\psi_L\psi_R + W^{*\prime\prime}(\Phi^*)\psi^*_L\psi^*_R\right]$$

bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu\cos\Phi\ , \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)$$

$$\psi_L = \nu + \imath\rho\,, \qquad \psi_R = -\beta + \imath\gamma$$

2d supersymmetry will be manifest in the S-matrix

2d susy in PR models for $AdS_3 \times S^3$ and $AdS_5 \times S^5$?

non-standard 2d susy conjectured: remnant of κ-symmetry
[Grigoriev, A.T. 97]
found recently: non-local susy
[Goykhman,Ivanov; Hollowood, Miramontes 2011]

(4,4) susy in $AdS_3 \times S^3$; (8,8) susy in $AdS_5 \times S^5$
"left" (8,0) part:

$$\delta_{\epsilon_L} g = g([T, [\Psi_R, \epsilon_L]] + \delta u)$$

$$\delta_{\epsilon_L} \Psi_R = [(g^{-1} D_+ g)^{||}, \epsilon_L] + [\Psi_R, \ \delta u]$$

$$\delta_{\epsilon_L} \Psi_L = \mu[T, \, g\epsilon_L g^{-1}] \, , \qquad \delta_{\epsilon_L} A_\pm = 0$$

$$\delta u = \mu(D_-)^{-1} \, [\epsilon_L, (g^{-1} \Psi_L g)^{\perp}]$$

meaning of non-locality? need extra auxiliary d.o.f.?

implications for S-matrix?
find quantum-deformed supersymmetry in the S-matrix
[Hoare, A.T., 2011]

Global symmetries

• 2d Poincare $\mathfrak{so}(1,1) \Subset \mathbb{R}^{1,1}$
• in string theory: part of $\widehat{\mathfrak{f}}$ left after choosing matrix T
(cf. choice of BMN vacuum)

$$\widehat{\mathrm{f}} = \widehat{\mathrm{f}}^{\perp} \oplus \widehat{\mathrm{f}}^{\parallel} \,, \qquad [\widehat{\mathrm{f}}^{\perp}, T] = 0$$

$$\widehat{\mathrm{f}}^{\perp} = \widehat{\mathfrak{h}} \oplus \{T\} \,, \qquad \widehat{\mathfrak{h}} = \mathfrak{h} \oplus \widehat{\mathfrak{f}}_1^{\perp} \oplus \widehat{\mathfrak{f}}_3^{\perp} \,, \qquad \mathfrak{h} = \widehat{\mathfrak{f}}_0^{\perp}$$

hidden symmetry of PR theory?
• $\widehat{\mathfrak{h}}$= R-symmetry+fermionic part of 2d susy algebra:

$$\mathfrak{s} = \mathfrak{so}(1,1) \Subset \left(\widehat{\mathfrak{h}} \ltimes \mathbb{R}^{1,1}\right)$$

• 2d susy originates from target space/κ susy of string theory
PR: target space susy Q's become "charged" under 2d Lorentz
– become generators of 2-d susy of PR theory
• 2d susy not manifest in the action beyond quadratic level:
realized non-locally (locally in $AdS_2 \times S^2$ case)
appears as quantum-deformed $U_q(\mathfrak{s})$ symmetry
of the perturbative S-matrix $\left(q = \exp(-i\frac{\pi}{k})\right)$

$AdS_2 \times S^2$:

$\widehat{\mathfrak{h}} = \mathfrak{psu}(1|1) \oplus \mathfrak{psu}(1|1)$

$\mathfrak{s}$ equivalent to (2,2) susy algebra in 2d

no quantum deformation

$AdS_3 \times S^3$:

$\widehat{\mathfrak{h}} = \left[\mathfrak{u}(1) \Subset \mathfrak{psu}(1|1) \oplus \mathfrak{psu}(1|1)\right]^{\oplus 2} \ltimes \mathfrak{u}(1)$

$\mathfrak{s}$ like (4,4) susy algebra in 2d

quantum-deformed symmetry of S-matrix

$AdS_5 \times S^5$:

$\widehat{\mathfrak{h}} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$

$\mathfrak{s}$ like (8,8) susy algebra in 2d

quantum-deformed symmetry of S-matrix

Quantum PR theory

Reduction procedure may work at quantum level
only in conformally invariant case (like $AdS_5 \times S^5$ string)
Consistency requires that reduced theory is also UV finite

gWZW + free fermions is finite;
due to fermions μ is not renormalized: remains arbitrary
conformal symmetry gauge fixing parameter at quantum level
[Roiban, A.T., 2009]

Thus reduced model is 2d Lorentz invariant
and power counting renormalizable – in fact, finite
(cf. l.c. gauge fixed GS superstring)

Relation of reduced theory and string theory at quantum level?

compare quantum partition functions

One-loop partition function:

semiclassical expansion near counterparts
of rigid strings in $AdS_5 \times S^5$ leads to same
characteristic frequencies – same 1-loop partition function
[Iwashita, Hoare, A.T. 09]

$$Z^{(1)}_{PR} = Z^{(1)}_{string}$$

one-loop matching is not too surprising
given classical equivalence but is still non-trivial:
due to standard kinetic terms in reduced theory, etc.
[not any two classically equivalent theories
will have same 1-loop partition functions]

Long folded (S, J) spinning string $\quad (m \sim \ln S,\ \mu \sim J)$

$$Y_0 + iY_5 = \cosh(m\sigma)\, e^{i\kappa\tau}, \quad Y_1 + iY_2 = \sinh(m\sigma)\, e^{i\kappa\tau}$$

$$X_1 + iX_2 = e^{i\mu\tau}, \qquad \kappa^2 = m^2 + \mu^2$$

corresponding PR solution:

in $AdS_3 \times S^1$

$$L = (\partial\phi)^2 + \coth^2\phi\,(\partial\chi)^2 - \tfrac{1}{2}\mu^2 \cosh 2\phi$$

$$\phi = \ln \frac{\kappa+m}{\mu}\ , \qquad \chi = -\frac{m}{\mu}\sigma$$

in $AdS_5 \times S^5$

$$g = \begin{pmatrix} 0 & \frac{\kappa}{\mu}v & -\frac{m}{\mu}v & 0 \\ -\frac{\kappa}{\mu}v^* & 0 & 0 & \frac{m}{\mu}v^* \\ \frac{m}{\mu}v & 0 & 0 & -\frac{\kappa}{\mu}v \\ 0 & -\frac{m}{\mu}v^* & \frac{\kappa}{\mu}v^* & 0 \end{pmatrix}, \qquad v = e^{-i\frac{\kappa^2\tau}{\mu}}$$

$$A_+ = \frac{i(m^2+\kappa^2)}{2\mu}\mathrm{diag}(1,-1,1,-1)$$

$$A_- = \frac{i\mu}{2}\mathrm{diag}(1,-1,1,-1)$$

same fluctuations as in string case –

same 1-loop partition function: $Z^{(1)}_{PR} = Z^{(1)}_{string}$

$\mu \to 0$ limit (rescaled by κ^2):

$$m^2_{AdS_3} = 4 \,, \qquad 2 \times m^2_{AdS_5} = 2$$
$$5 \times m^2_{S_5} = 0 \,, \qquad 8 \times m^2_F = 1$$

String partition function: ($f_{tot} = \sqrt{\lambda} + f$)

$$\Gamma = -\ln Z = \frac{1}{2\pi} f(\lambda) \kappa^2 V_2$$

$$f(\lambda) = a_1 + \frac{a_2}{\sqrt{\lambda}} + O(\frac{1}{(\sqrt{\lambda})^2})$$

$$a_1 = -3 \ln 2 \,, \qquad a_2 = -K$$

String theory 2-loop correction :

$$a_2 = a_{2B} + a_{2F} = K - 2K = -K$$

Catalan's constant comes from sunset integrals
with AdS_5 modes transverse to AdS_3 (i.e. $m^2_{AdS_5} = 2$)
[Roiban, Tirziu, A.T., 2007]

$$I[m_1^2, m_1^2, m_1^2] \equiv \int \frac{d^2p_1 d^2p_2 d^2p_3 \ \delta^{(2)}(p_1 + p_2 + p_3)}{(p_1^2 + m_1^2)(p_2^2 + m_2^2)(p_3^2 + m_3^2)}$$

$$I[4,2,2] = \frac{1}{(4\pi)^2} K \ , \qquad I[2,1,1] = -\frac{2}{(4\pi)^2} K$$

K-terms thus absent in $AdS_3 \times S^3$ case
[Iwashita, Roiban, A.T.]

$$AdS_3 \times S^3 : \qquad a_1 = -2 \ln 2 \ , \qquad a_2 = 0$$

Reduced theory 2-loop correction :

similar 2-loop computation gives (k as coupling constant)

$$\widetilde{\Gamma} = -\ln Z_{PR} = \frac{1}{2\pi} \widetilde{f}(\lambda) \kappa^2 V_2$$

$$\widetilde{f}(\lambda) = \widetilde{a}_1 + \frac{2\widetilde{a}_2}{k} + O(\frac{1}{k^2})$$

$AdS_3 \times S^3$ case:

$$\widetilde{a}_1 = -2\ln 2 \ , \qquad \widetilde{a}_2 = -(\ln 2)^2$$

if $k = 2\sqrt{\lambda}$ this implies

$$\widetilde{a}_1 = a_1 \ , \qquad \widetilde{a}_2 = a_2 - \frac{1}{4} a_1^2$$

string and PR partition functions are closely related

$AdS_5 \times S^5$ case:

$$\widetilde{a}_1 = -3\ln 2 = a_1 \ ,$$

$$\widetilde{a}_2 = -K - \frac{9}{4}(\ln 2)^2 = a_2 - \frac{1}{4}a_1^2$$

K-terms match if $k = 2\sqrt{\lambda}$
same pattern of K contributions as in string theory:
come from similar integrals
bosons $\to +K$, fermions $\to -2K$
again get

$$\widetilde{a}_2 = a_2 - \frac{1}{4}a_1^2$$

nontrivial: no other structures like $I[4,4,4]$, etc.
matching of K-terms is remarkable
suggests close relation between two quantum theories

precise relation between quantum partition functions?
explanation for $- - \frac{1}{4}a_1^2$?

$k = 2\sqrt{\lambda}$?

compare classical actions:

$$I_{string} = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \ \mathrm{Str}(P_+ P_- + ...)$$

$$I_{\mathrm{PR}} = \frac{k}{8\pi} \int d^2\sigma \ \mathrm{Str}\Big[\tfrac{1}{2}(g^{-1}\partial g)^2 + ... + \mu^2 g^{-1} T g T + ...\Big]$$

since $P_+ = \mu T \ , \quad P_- = \mu g^{-1} T g$

potential plus Yukawa terms = superstring action

suggests identification $k = 2\sqrt{\lambda}$

k should not be quantized?

[different boundary conditions/solitons in massive theory

as compared to standard massless gWZW model?]

S-matrix for elementary excitations

Step towards exact solution: S-matrix
Integrable theory – determined by 2-particle S-matrix
expand action around trivial vacuum
$g = \mathbf{1},\ A_\pm = 0,\ \Psi_{_R} = \Psi_{_L} = 0$
find two-particle scattering amplitude
for the 8+8 elementary massive excitations

$$g = e^{\eta}\,, \qquad \eta \in \mathfrak{g}$$

decompose η into coset (“physical”) and subgroup (“gauge”) parts

$$\eta = X + \xi\,, \qquad X \in \mathfrak{m}\,, \qquad \xi \in \mathfrak{h}$$

$A_+ = 0$ gauge: preserves 2d Lorentz inv
Integrate over A_-: delta-function constraint on ξ

$$\partial_+\xi - \frac{1}{2}[X,\, \partial_+ X] - \frac{1}{2}[\xi,\, \partial_+\xi] + \ldots = 0$$

solving for ξ gives action for physical d.o.f. $(X,\ \Psi_R,\ \Psi_L)$

$$
\begin{aligned}
\widetilde{L} = \frac{k}{4\pi}\ \mathrm{STr}\Big(\ & \frac{1}{2}\partial_+ X\partial_- X - \frac{\mu^2}{2}X^2 \\
& +\Psi_L T\partial_+\Psi_L + \Psi_R T\partial_-\Psi_R + \mu\Psi_L\Psi_R \\
& +\frac{1}{12}[X,\ \partial_+ X][X,\ \partial_- X] + \frac{\mu^2}{24}[X,\ [X,\ T]]^2 \\
& -\frac{1}{4}[\Psi_L T,\ \Psi_L][X,\ \partial_+ X] - \frac{1}{4}[\Psi_R,\ T\Psi_R][X,\ \partial_- X] \\
& -\frac{\mu}{2}[X,\ \Psi_R][X,\ \Psi_L] + \frac{1}{2}[\Psi_L T,\ \Psi_L][\Psi_R,\ T\Psi_R] + \ldots\Big)
\end{aligned}
$$

remaining symmetry: global part o gauge group H

$$
(X,\ \Psi_R,\ \Psi_L) \to h^{-1}(X,\ \Psi_R,\ \Psi_L)h
$$

basic fields $X = Y \oplus Z,\ \Psi = \zeta \oplus \chi$ in 8×8 matrix

$$\begin{pmatrix} SU(2)_1 & Y & 0 & \zeta \\ Y & SU(2)_{\dot{1}} & \chi & 0 \\ 0 & \chi & SU(2)_2 & Z \\ \zeta & 0 & Z & SU(2)_{\dot{2}} \end{pmatrix}$$

introduce bosonic (a, $\dot{a}$) and fermionic (α, $\dot{\alpha}$) indices= 1,2:

$SU(2)_1$: a $\quad SU(2)_2$: α $\quad SU(2)_{\dot{1}}$: $\dot{a}$ $\quad SU(2)_{\dot{2}}$: $\dot{\alpha}$

$$L = \partial_+ Y_{a\dot{a}} \partial_- Y^{\dot{a}a} - \mu^2 Y_{a\dot{a}} Y^{\dot{a}a}$$

$$+ \partial_+ Z_{\alpha\dot{\alpha}} \partial_- Z^{\dot{\alpha}\alpha} - \mu^2 Z_{\alpha\dot{\alpha}} Z^{\dot{\alpha}\alpha}$$

$$+ i\zeta_{L\,a\dot{\alpha}} \partial_+ \zeta_L{}^{\dot{\alpha}a} + i\zeta_{R\,a\dot{\alpha}} \partial_- \zeta_R{}^{\dot{\alpha}a} - 2i\mu \zeta_{L\,a\dot{\alpha}} \zeta_R{}^{\dot{\alpha}a}$$

$$+ i\chi_{L\,\alpha\dot{a}} \partial_+ \chi_L{}^{\dot{a}\alpha} + i\chi_{R\,\alpha\dot{a}} \partial_- \chi_R{}^{\dot{a}\alpha} - 2i\mu \chi_{L\,\alpha\dot{a}} \chi_R{}^{\dot{a}\alpha}$$

$$- \frac{2\pi}{3k} \left(Y_{a\dot{a}} Y^{\dot{a}a} \partial_+ Y_{b\dot{b}} \partial_- Y^{\dot{b}b} - Y_{a\dot{a}} \partial_+ Y^{\dot{a}a} Y_{b\dot{b}} \partial_- Y^{\dot{b}b} \right) + \ldots$$

combine $Y_{a\dot{a}}$, $Z_{\alpha\dot{\alpha}}$, $\zeta_{a\dot{\alpha}}$, $\chi_{\alpha\dot{a}}$ into

$$\Phi_{A\dot{A}}\,, \qquad\qquad A = (a, \alpha)$$

S-matrix acting on 2-particle state:

$$\mathbb{S}\,|\Phi_{A\dot{A}}(\vartheta_1)\Phi_{B\dot{B}}(\vartheta_2)\rangle = S_{A\dot{A},B\dot{B}}^{C\dot{C},D\dot{D}}(\theta,k)\,|\Phi_{C\dot{C}}(\vartheta_1)\Phi_{D\dot{D}}(\vartheta_2)\rangle$$

Lorentz invariance: two-particle S-matrix depends on
$\theta = \vartheta_1 - \vartheta_2, \qquad p_{i\,0} = \mu\cosh\vartheta_i, \qquad p_{i\,1} = \mu\sinh\vartheta_i$

Remarkably, resulting S-matrix group-factorizes :

$$S_{A\dot{A},B\dot{B}}^{C\dot{C},D\dot{D}}(\theta,k) = (-1)^{[B][\dot{A}]+[D][\dot{C}]}S_{AB}^{CD}(\theta,k)S_{\dot{A}\dot{B}}^{\dot{C}\dot{D}}(\theta,k)$$

• generic integrable theory with $G_1 \times G_2$ symmetry
and fields in bi-fundamental representation:
S-matrix should group-factorize

• happens in l.c. gauge $AdS_5 \times S^5$ superstring S-matrix
invariant under product supergroup $PSU(2|2) \times PSU(2|2)$
[Kloze,MacLoughlin,Roiban,Zarembo 06;
Arutyunov,Frolov,Zamaklar 06]

• field contents of l.c. superstring and reduced theory
are identical w.r.t. bosonic symmetry $[SU(2)]^4$
superstring: integrability and $PSU(2|2)\times PSU(2|2)$ symmetry

• PR model: integrability but no manifest supersymmetry;
perturbative factorization suggests hidden supergroup symmetry

S-matrix : 10 functions $K_n(\theta, k)$

$$S_{AB}^{CD}(\theta, k) = \begin{cases} K_1(\theta,k)\ \delta_a^c\delta_b^d + K_2(\theta,k)\ \delta_a^d\delta_b^c\,, \\ K_3(\theta,k)\ \delta_\alpha^\gamma\delta_\beta^\delta + K_4(\theta,k)\ \delta_\alpha^\delta\delta_\beta^\gamma\,, \\ K_5(\theta,k)\ \epsilon_{ab}\epsilon^{\gamma\delta}\,, \qquad K_6(\theta,k)\ \epsilon_{\alpha\beta}\epsilon^{cd}\,, \\ K_7(\theta,k)\ \delta_a^d\delta_\beta^\gamma\,, \qquad K_8(\theta,k)\delta_\alpha^\delta\delta_b^c\,, \\ K_9(\theta,k)\ \delta_a^c\delta_\beta^\delta\,, \qquad K_{10}(\theta,k)\ \delta_\alpha^\gamma\delta_b^d\,, \end{cases}$$

$$K_1(\theta,k) = K_3(\theta,-k) = 1 + \frac{i\pi}{2k}\tanh\frac{\theta}{2} + \mathcal{O}(\frac{1}{k^2})$$

$$K_2(\theta,k) = K_4(\theta,-k) = -\frac{i\pi}{k}\coth\theta + \mathcal{O}(\frac{1}{k^2})$$

$$K_5(\theta,k) = -K_6(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} + \mathcal{O}(\frac{1}{k^2})$$

$$K_7(\theta,k) = -K_8(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} + \mathcal{O}(\frac{1}{k^2})$$

$$K_9(\theta,k) = K_{10}(\theta,-k) = 1 + \mathcal{O}(\frac{1}{k^2})$$

compare to l.c. gauge tree-level $AdS_5 \times S^5$ string S-matrix :

$\bar{K}_n \equiv (K_n)_{string}$ depend separately on 2 rapidities
and $\frac{1}{k} \to \frac{1}{\sqrt{\lambda}}$

$$
\begin{aligned}
\bar{K}_{1,3} &= 1 \pm \tfrac{2\pi}{\sqrt{\lambda}}(\sinh\vartheta_1 - \sinh\vartheta_2)^2 + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^2}) \\
\bar{K}_{2,4} &= \pm\tfrac{8\pi}{\sqrt{\lambda}}\sinh\vartheta_1 \sinh\vartheta_2 + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^2}) \\
\bar{K}_{5,6} &= \tfrac{8\pi}{\sqrt{\lambda}}\sinh\vartheta_1 \sinh\vartheta_2 \sinh\tfrac{\vartheta_1-\vartheta_2}{2} + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^2}) \\
\bar{K}_{7,8} &= \tfrac{8\pi}{\sqrt{\lambda}}\sinh\vartheta_1 \sinh\vartheta_2 \cosh\tfrac{\vartheta_1-\vartheta_2}{2} + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^2}) \\
\bar{K}_{9,10} &= 1 \mp \tfrac{2\pi}{\sqrt{\lambda}}(\sinh^2\vartheta_1 + \sinh^2\vartheta_2) + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^2})
\end{aligned}
$$

Tree-level S-matrix of $AdS_5 \times S^5$ PR model :

- unitary and crossing-symmetric
- satisfies group factorisation, but not Yang-Baxter equation:

clash between relativistic invariance, trigonometric structure
and manifest non-abelian symmetry $H = [SU(2)]^4$
(string S-matrix is not Lorentz inv. but does satisfy YBE)

- K_n are same as in q-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$

R-matrix of quantum-deformed Hubbard model
[Beisert, Koroteev, 2008; Beisert, 2010]

- suggests that $SU(2) \times SU(2)$ symmetry

should be quantum-deformed rather than manifest

One-loop correction to S-matrix

1-loop corrections to 2-particle scattering from quartic Lagrangian:
standard massive 2d Feynman graphs [Hoare, AAT, 2011]

$$K_n = \Phi_0(\theta,k)\,\widehat{K}_i(\theta,k)$$

$$\widehat{K}_1(\theta,k) = \widehat{K}_3(\theta,-k) = 1 + \tfrac{i\pi}{2k}\tanh\tfrac{\theta}{2} - \tfrac{5\pi^2}{8k^2} - \tfrac{i\pi\theta}{2k^2} + \mathcal{O}(\tfrac{1}{k^3})$$
$$\widehat{K}_2(\theta,k) = \widehat{K}_4(\theta,-k) = -\tfrac{i\pi}{k}\coth\theta + \tfrac{\pi^2}{2k^2} + \tfrac{i\pi\theta}{k^2} + \mathcal{O}(\tfrac{1}{k^3})$$
$$\widehat{K}_5(\theta,k) = -\widehat{K}_6(\theta,-k) = -\tfrac{i\pi}{2k}\,\mathrm{sech}\,\tfrac{\theta}{2} + \mathcal{O}(\tfrac{1}{k^3})$$
$$\widehat{K}_7(\theta,k) = -\widehat{K}_8(\theta,-k) = -\tfrac{i\pi}{2k}\,\mathrm{cosech}\,\tfrac{\theta}{2} + \mathcal{O}(\tfrac{1}{k^3})$$
$$\widehat{K}_9(\theta,k) = \widehat{K}_{10}(\theta,-k) = 1 + \mathcal{O}(\tfrac{1}{k^3})$$
$$\Phi_0 = 1 + \tfrac{\pi\,\mathrm{cosech}\,\theta}{4k^2}\Big(i\big[2+(i\pi-2\theta)\coth\theta\big] - \pi\,\mathrm{cosech}\,\theta\Big) + \mathcal{O}(\tfrac{1}{k^3})$$

to get idea of how to interpret/generalize this S-matrix study
special cases/truncations :
PR models for $AdS_2\times S^2$ and $AdS_3\times S^3$

$AdS_2 \times S^2$ case

PR model equivalent to $\mathcal{N}=2$ supersymmetric sine-Gordon
tree + 1-loop corrections agree with expansion of known
exact S-matrix of $\mathcal{N}=2$ susy SG
[Kobayashi, Uematsu 91; Ahn 91; Shankar, Witten 78]

$$S_{sg}(\theta,k)\otimes S_1(\theta,k)\otimes S_1(\theta,k)$$

$$S_{sg}=\frac{\sinh\theta+i\sin\frac{\pi}{k}}{\sinh\theta-i\sin\frac{\pi}{k}}$$

$$S_1\sim\frac{\sinh\theta-i\sin\frac{\pi}{k}}{\sinh\theta+i\sin\frac{\pi}{k}}\,Y(\theta,k)\,Y(i\pi-\theta,k)$$

$$Y=\prod_{l=1}^{\infty}\frac{\Gamma\left(\frac{1}{2k}-\frac{i\theta}{2\pi}+l\right)\Gamma\left(-\frac{1}{2k}-\frac{i\theta}{2\pi}+l-1\right)\Gamma\left(-\frac{i\theta}{2\pi}+l-\frac{1}{2}\right)\Gamma\left(-\frac{i\theta}{2\pi}+l+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2k}-\frac{i\theta}{2\pi}+l+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2k}-\frac{i\theta}{2\pi}+l-\frac{1}{2}\right)\Gamma\left(-\frac{i\theta}{2\pi}+l-1\right)\Gamma\left(-\frac{i\theta}{2\pi}+l\right)}$$

manifestly invariant under (2,2) susy
which in PR model framework is interpreted as

$$\mathfrak{so}(1,1)\Subset(\widehat{\mathfrak{f}}^{\perp}\ltimes\mathbb{R}^{1,1})\,,\qquad\qquad \widehat{\mathfrak{f}}^{\perp}=\mathfrak{psu}(1|1)\oplus\mathfrak{psu}(1|1)$$

$AdS_3 \times S^3$ case

• here $a, \dot{a}, \alpha, \dot{\alpha}$ are vector $SO(2)$ indices
4+4 fields $Y_{a\dot{a}},\ Z_{\alpha\dot{\alpha}},\ \zeta_{a\dot{\alpha}},\ \chi_{\alpha\dot{a}}$ (with $Y_{a\dot{a}} = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}Y_{b\dot{b}}$, etc.)
can again be packaged into single $\Phi_{A\dot{A}}$
• S-matrix again group-factorizes
S_{AB}^{CD} expressed in terms of 12 functions $L_n(\theta, k)$
with similar tree ($\frac{1}{k}$) and 1-loop ($\frac{1}{k^2}$) terms
• $H = U(1) \times U(1)$ invariant S-matrix satisfies YBE

• Supersymmetry ?
by analogy with $AdS_2 \times S^2$ case
conjecture that it is determined by $\widehat{\mathfrak{f}}^{\perp}$

$$\mathfrak{so}(1,1) \Subset (\mathfrak{t} \oplus \mathfrak{t} \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^{1,1}) \,, \qquad \mathfrak{t} = \mathfrak{u}(1) \Subset \mathfrak{psu}(1|1)$$

susy : $\mathfrak{k} \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^{1,1}$; should act on factor S-matrix S_{AB}^{CD}

$$
\begin{aligned}
&[\mathfrak{R},\, \mathfrak{R}] = 0\,, && [\mathfrak{L},\, \mathfrak{L}] = 0\,,\\
&[\mathfrak{R},\, \mathfrak{Q}_{\pm\mp}] = \pm i\mathfrak{Q}_{\pm\mp}\,, && [\mathfrak{L},\, \mathfrak{Q}_{\pm\mp}] = \mp i\mathfrak{Q}_{\pm\mp}\,,\\
&[\mathfrak{R},\, \mathfrak{S}_{\pm\mp}] = \pm i\mathfrak{S}_{\pm\mp}\,, && [\mathfrak{L},\, \mathfrak{S}_{\pm\mp}] = \mp i\mathfrak{S}_{\pm\mp}\,,\\
&\{\mathfrak{S}_{\pm\mp},\, \mathfrak{Q}_{\pm\mp}\} = 0\,, && \{\mathfrak{S}_{\pm\mp},\, \mathfrak{Q}_{\mp\pm}\} = \pm\tfrac{i}{2}(\mathfrak{R}+\mathfrak{L}) \equiv \pm\mathfrak{A}\,,\\
&\{\mathfrak{Q}_{\pm\mp},\, \mathfrak{Q}_{\pm\mp}\} = 0\,, && \{\mathfrak{Q}_{\pm\mp},\, \mathfrak{Q}_{\mp\pm}\} = -\mathfrak{P}_{+}\,,\\
&\{\mathfrak{S}_{\pm\mp},\, \mathfrak{S}_{\pm\mp}\} = 0\,, && \{\mathfrak{S}_{\pm\mp},\, \mathfrak{S}_{\mp\pm}\} = \mathfrak{P}_{-}
\end{aligned}
$$

$\mathfrak{R}$ and $\mathfrak{L}$: bosonic $u(1) \oplus u(1)$ generators
$\mathfrak{Q}_{\pm\mp}/\mathfrak{S}_{\pm\mp}$: 2+2 positive/negative chirality supercharges
$\mathfrak{P}_{+},\ \mathfrak{P}_{+}$: 2 central extensions – 2-d momenta

This is not manifest symmetry of 1-loop S-matrix but
quantum-deformed :

$$
\{\mathfrak{S}_{\pm\mp},\, \mathfrak{Q}_{\mp\pm}\} = \pm[\mathfrak{A}]_q\,, \qquad q = e^{-i\frac{2\pi}{k}}
$$

$$
[\mathfrak{A}]_q \equiv \frac{q^{\mathfrak{A}} - q^{-\mathfrak{A}}}{q - q^{-1}} = \mathfrak{A} + \frac{2\pi^2}{3k^2}(\mathfrak{A} - \mathfrak{A}^3) + \ldots
$$

Action of symmetry on 2-particle states: coproduct
should respect commutation relations – if deform the algebra need replace standard Leibnitz rule

$$\Delta(\mathfrak{J}) = \mathbb{I} \otimes \mathfrak{J} + \mathfrak{J} \otimes \mathbb{I}$$

by deformed one for action of fermionic generators
(abelian bosonic part not deformed):

$$\Delta(\mathfrak{Q}_{\pm\mp}) = \mathfrak{Q}_{\pm\mp} \otimes q^{-\mathfrak{A}} + \mathbb{I} \otimes \mathfrak{Q}_{\pm\mp}$$
$$\Delta(\mathfrak{S}_{\pm\mp}) = \mathfrak{S}_{\pm\mp} \otimes \mathbb{I} + q^{\mathfrak{A}} \otimes \mathfrak{S}_{\pm\mp}$$

Now use
(i) analogy with (2,2) supersymmetric $AdS_2 \times S^2$ case
(ii) analogy with complex SG S-matrix
(iii) explicit tree-level +1-loop data
to conjecture exact (in $1/k$) S-matrix
for elementary excitations of $AdS_3 \times S^3$ PR model

Exact S-matrix of $AdS_3 \times S^3$ PR model :

- assume q-deformed (4,4) supersymmetry is exact symmetry
- fix phase factor from unitarity, crossing and 1-loop data

$$L_{1,3} = \frac{1}{2}\left[P_1(\theta,k)\frac{\cosh\left(\frac{\theta}{2}\pm\frac{i\pi}{k}\right)}{\cosh\frac{\theta}{2}} + P_2(\theta,k)\frac{\sinh\left(\frac{\theta}{2}\mp\frac{i\pi}{k}\right)}{\sinh\frac{\theta}{2}}\right]$$

$$L_{2,4} = \frac{1}{2}\left[P_1(\theta,k)\frac{\cosh\left(\frac{\theta}{2}\pm\frac{i\pi}{k}\right)}{\cosh\frac{\theta}{2}} - P_2(\theta,k)\frac{\sinh\left(\frac{\theta}{2}\mp\frac{i\pi}{k}\right)}{\sinh\frac{\theta}{2}}\right]$$

$$L_{5,7}, L_{6,8} = \frac{1}{2}\Big[P_1(\theta,k) \pm P_2(\theta,k)\Big]$$

$$L_{9,10} = \frac{i}{2}P_1(\theta,k)\frac{\sin\frac{\pi}{k}}{\cosh\frac{\theta}{2}}\,, \qquad L_{11,12} = -\frac{i}{2}P_2(\theta,k)\frac{\sin\frac{\pi}{k}}{\cosh\frac{\theta}{2}}$$

$$P_1 = \sqrt{\frac{\cosh\left(\frac{\theta}{2}+\frac{i\pi}{k}\right)}{\cosh\left(\frac{\theta}{2}-\frac{i\pi}{k}\right)}}\prod_{l=1}^{\infty}\frac{\Gamma(\frac{i\theta}{2\pi}-\frac{1}{k}+l-\frac{1}{2})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{k}+l+\frac{1}{2})}{\Gamma(-\frac{i\theta}{2\pi}-\frac{1}{k}+l-\frac{1}{2})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{k}+l+\frac{1}{2})}$$

$$\times\frac{\Gamma(-\frac{i\theta}{2\pi}+l-\frac{1}{2})\Gamma(-\frac{i\theta}{2\pi}+l+\frac{1}{2})}{\Gamma(\frac{i\theta}{2\pi}+l-\frac{1}{2})\Gamma(\frac{i\theta}{2\pi}+l+\frac{1}{2})}$$

$$P_2(\theta,k) = P_1(i\pi-\theta,k)$$

What does this suggest about S-matrix of $AdS_5 \times S^5$ PR theory?

• perturbative tree + 1-loop S-matrix from Lagrangian theory has bosonic $H = [SU(2)]^4$ symmetry: does not satisfy YBE
– some subtlety in how integrability is realised?
• this perturbative S-matrix closely related (by a rotation) to S-matrix satisfying YBE with H broken/deformed ?

• analogy with $AdS_2 \times S^2$ and $AdS_3 \times S^3$ suggests to expect quantum-deformed (8,8) supersymmetry related to $\mathfrak{f}^\perp$:

$$\mathfrak{so}(1,1) \Subset (\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \ltimes \mathbb{R}^2)$$

bosonic part $[su(2)]^{\oplus 4}$ is also quantum-deformed
• in similar bosonic G/H theories with non-abelian H conjectured soliton S-matrix has q-deformed symmetry [Hollowood, Miramontes, 2009-11]
$q = e^{-i\frac{\pi}{k}}$ -deformation appears in WZW-related contexts

Remarkably, exists relativistic S-matrix with such q-deformed supersymmetry, and it satisfies YBE [Hoare, A.T. 2011]:

given by a trigonometric relativistic limit of 2-parameter q-deformed $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ R-matrix constructed by Beisert and Koroteev, 2008; Beisert 2010:

$$g \to \infty, \qquad\qquad q = e^{-i\frac{\pi}{k}}$$

S-matrix depends on θ and single parameter k
relation to Lagrangian-theory S-matrix supported by
close connection at tree level:
same coefficients, two S-matrices are related by a rotation

natural candidate for exact S-matrix of $AdS_5 \times S^5$ PR theory

Exact S-matrix of $AdS_5 \times S^5$ PR model?

Structure similar to S-matrix in $AdS_3 \times S^3$ case
with 10 coefficient functions $J_n(\theta, k)$ given by

$$J_{1,3} = P_0(\theta,k) \cos\tfrac{\pi}{k}\,\mathrm{sech}\,\tfrac{\theta}{2}\,\cosh\left(\tfrac{\theta}{2} \pm \tfrac{i\pi}{2k}\right)$$

$$J_{2,4} = \mp i P_0(\theta,k)\left[1 - \cos\tfrac{\pi}{k} + \cosh\theta + \cosh\left(\theta \pm \tfrac{i\pi}{k}\right)\right]\frac{\sin\frac{\pi}{2k}}{\sinh\theta}$$

$$J_{5,6} = -i P_0(\theta,k) \cos\tfrac{\pi}{k}\,\sin\tfrac{\pi}{2k}\,\mathrm{sech}\,\tfrac{\theta}{2}$$

$$J_{7,8} = -i P_0(\theta,k) \sin\tfrac{\pi}{2k}\,\mathrm{cosech}\,\tfrac{\theta}{2}\,, \qquad J_{9,10} = P_0(\theta,k)$$

$$P_0(\theta,k) = \sqrt{\frac{\sinh\theta - i\sin\frac{\pi}{k}}{\sinh\theta + i\sin\frac{\pi}{k}}}\;Y(\theta,k)\,Y(i\pi-\theta,k)$$

$$Y(\theta,k) = \prod_{l=1}^{\infty} \frac{\Gamma\left(\frac{1}{2k}-\frac{i\theta}{2\pi}+l\right)\Gamma\left(-\frac{1}{2k}-\frac{i\theta}{2\pi}+l-1\right)}{\Gamma\left(\frac{1}{2k}-\frac{i\theta}{2\pi}+l+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2k}-\frac{i\theta}{2\pi}+l-\frac{1}{2}\right)} \times \frac{\Gamma\left(-\frac{i\theta}{2\pi}+l-\frac{1}{2}\right)\Gamma\left(-\frac{i\theta}{2\pi}+l+\frac{1}{2}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}+l-1\right)\Gamma\left(-\frac{i\theta}{2\pi}+l\right)}$$

supported by relation via fusion/bootstrap procedure
to spectrum of solitons found by Hollowood and Miramontes
[Hoare, Hollowood, Miramontes 2011]

Conclusions

- PR model: special relativistic massive integrable finite model
  closely related to $AdS_5 \times S^5$ superstring: classical equivalence
  1-loop partition functions for classical solutions match;
  2-loop partition function for infinite spin limit of folded string:
  non-trivial – Catalan's constant – part matches string result
  suggests relation between quantum PR and string theories

- S-matrix for perturbative excitations of PR theory:
  relativistic "analog" of magnon S-matrix in string theory
  PR $AdS_2 \times S^2$: equivalent to $(2,2)$ susy sine-Gordon theory
  PR $AdS_3 \times S^3$: fermionic generalisazation of CSG + CShG
  S-matrix has novel q-deformed (4,4) 2d susy
  PR $AdS_5 \times S^5$: candidate for exact S-matrix with q-deformed
  (8,8) susy: $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \ltimes \mathbb{R}^2$

Open questions

- deeper understanding of relation between string and PR theory
  at classical level: meaning of novel 2d susy? ...

- precise relation of quantum string and quantum PR theory?
  $k \sim \sqrt{\lambda}$? relation between quantum partition functions?
  relation between S-matrices?

- origin of quantum deformation from Lagrangian point of view?
  q-deformed susy: relation to classical non-local 2d susy?
  reason for q-deformation of global $H = [SU(2)]^4$ symmetry?

- exact relation between perturbative H-symmetric S-matrix
  and YBE-satisfying S-matrix with q-deformed symmetry?

How understanding of PR theory helps us
in solving quantum $AdS_5 \times S^5$ string from first principles?