Universal properties of two-dimensional percolation

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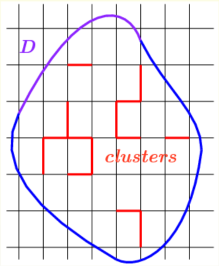
Percolation, definitions

- Random bond percolation is the problem of occupying the edges of a graph or of a lattice region D with probability p.



- **Nomenclature**

1. The set of occupied bonds is a graph $\mathcal{G}$
2. Connected components $c \subseteq \mathcal{G}$ are called clusters

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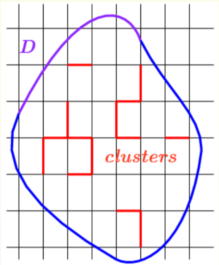

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- Graphs have factorized probability measure

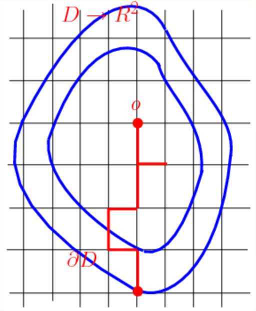
$$\mu(\mathcal{G}) = p^{\#\text{ bonds}}(1-p)^{\#\text{ empty bonds}}$$

- By definition the percolative partition function

$$Z_D = \sum_{\mathcal{G}\subseteq D} \mu(\mathcal{G}) = 1, \quad \text{in any domain } D.$$

Percolation as a critical phenomenon

- Paradigm for **geometric** phase transitions (no sp. symmetry breaking, no dynamical d.o.f.)



Order parameter

$$P = \frac{\#\text{sites in the ``infinite'' cluster}}{\#\ \text{sites}}$$
$$\sim (p - p_c)^\beta,\ \ p \to p_c^+$$

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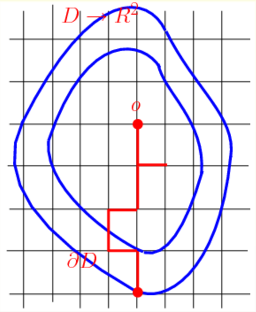

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- A bit of history of the exact (universal) results obtained at the critical point in two dimensions

80's-90's	Critical exponents (Dotsenko-Fateev, Nienhuis, Duplantier-Saleur,...)
90's-Present	Crossing probabilities, SLE (Cardy, Bauer-Bernard, Schramm, Smirnov,...)
Last ten years	Virasoro algebra rep. (Gurarie, Mathieu-Ridout, Pearce-Rasmussen-Zuber,...)

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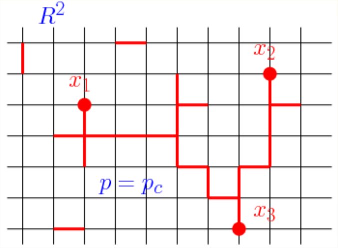

- Even at p_c, ***n*-point connectivities** for $n \geq 3$

$$P(x_1, \dots, x_n) = \text{Prob. } \{x_1, \dots, x_n\}$$
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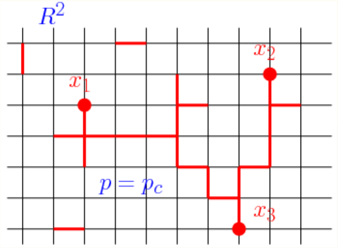

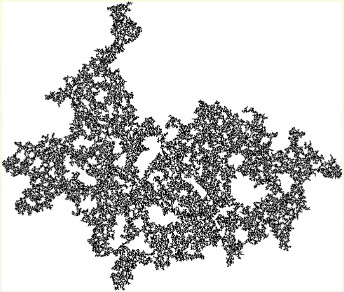

- Even at p_c, ***n*-point connectivities** for $n \geq 3$

$$P(x_1, \ldots, x_n) = \text{Prob. } \{x_1, \ldots, x_n\} \text{ are connected}$$

- **Off-critical quantities**, starting from the universal ratio

$$\frac{\Gamma^+}{\Gamma^-} = \frac{\text{size of finite cl. below } p_c}{\text{size of finite cl. above } p_c}$$

The Random Cluster Model (RCM)

- A field theory description of Percolation is obtained through the mapping with the ferromagnetic S_q **invariant**, q-color Potts model,

$$\mathcal{H}_{\text{Potts}} = -J \sum_{\langle x,y \rangle} \delta_{s(x),s(y)}; \quad s(x) = 1, \ldots, q \in \mathbb{N}.$$

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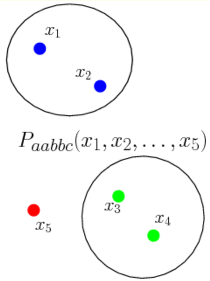
- Such mapping was proposed by Fortuin and Kasteleyn **(FK 70's)**

$$\begin{aligned} Z_{\text{Potts}} &= \sum_{\{s(x)\}} \mathrm{e}^{-\mathcal{H}_{\text{Potts}}} && \text{(expanding } \mathrm{e}^{J\delta}\text{)} \\ &= \sum_{\{s(x)\}} \prod_{\langle x,y \rangle} \left[(\mathrm{e}^J - 1)\delta_{s(x),s(y)} + 1 \right] && \text{(with } p = 1 - \mathrm{e}^{-J}\text{)} \\ &= \mathrm{e}^{NJ} \sum_{\mathcal{G} \subseteq L} \overbrace{q^{\#\text{clusters}} p^{\#\text{bonds}} (1-p)^{\#\text{empty bonds}}}^{\text{Graph prob. measure } \mu(q,\mathcal{G})} \equiv Z_{RCM}(q). \end{aligned}$$

- FK provides an **analytic continuation** of the Potts p.f. to $q \in \mathbb{R}$ and the RCM measure defines $(q > 0)$ a correlated percolation problem.

Some comments on the analytic continuation...

(G. Delfino and JV, NPB 2011)



- Observables in the RCM are ***n*-point connectivities** $P_{a_1 \ldots a_n}(x_1, \ldots, x_n)$
- Their full number $(p < p_c)$ is B_n, the number of **partitions** of a set of n elements.

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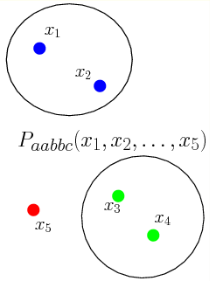

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- Their full number $(p < p_c)$ is B_n, the number of **partitions** of a set of n elements.

- The B_n n-point connectivities $P_{a_1,\ldots,a_n}(x_1, \ldots, x_n)$ are not linearly independent but satisfy sum rules

Examples

$$P_{aa}(x_1, x_2) + P_{ab}(x_1, x_2) = 1 \qquad n = 2$$

$$P_{aaa}(x_1, x_2, x_3) + P_{aab}(x_1, x_2, x_3) = P_{aa}(x_1, x_2) \qquad n = 3$$

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- Counting the number of linear relations we found that the number of independent n-point connectivities is the number F_n of **non-singleton partitions** of a set of n elements.

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- Counting the number of linear relations we found that the number of independent n-point connectivities is the number F_n of **non-singleton partitions** of a set of n elements.
- Remarkably, F_n is also the number of

1. Independent Potts spin n-point c.f. in the formal and unbroken S_q symmetric Potts field theory

$$\langle \sigma_{\alpha_1}(x_1) \dots \sigma_{\alpha_n}(x_n) \rangle, \quad \sum_{\alpha=1}^{q} \sigma_\alpha(x) = 0$$

2. In 2d the number of independent c.f. of the disorder field $\mu_{\alpha\beta}$, creating a domain wall $(\alpha - \beta)$ in the S_q sp. broken phase

$$\langle \mu_{\alpha_1\alpha_2}(x_1) \dots \mu_{\alpha_n\alpha_1}(x_n) \rangle \equiv \boxed{\alpha_1}\boxed{\alpha_2}\boxed{\alpha_3}\boxed{\dots}\boxed{\alpha_1}$$

Applications to 2d percolation, the 3-point connectivity (G. Delfino and JV, JPA 2011)

- Consider the three-point connectivity in 2d percolation at the percolation threshold $p = p_c$

$$
\begin{aligned}
P_{aaa} &= \lim_{q\to 1} \frac{1}{(q-1)(q-2)} \langle \sigma_\alpha(x_1)\sigma_\alpha(x_2)\sigma_\alpha(x_3)\rangle_{T=T_c} && \text{(FK mapping)} \\
&= \lim_{q\to 1} \langle \mu_{\alpha\beta}(x_1)\mu_{\beta\gamma}(x_2)\mu_{\gamma\alpha}(x_3)\rangle_{T=T_c} && \text{(Potts duality)} \\
&= \lim_{q\to 1} C_\mu(q)\sqrt{P_{aa}(x_1,x_2)P_{aa}(x_1,x_3)P_{aa}(x_2,x_3)} && \text{(Conformal inv.)}
\end{aligned}
$$

- The structure constant $C_\mu(q)$ appears in the S_q symmetric OPE of the Potts disorder fields

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\lim_{x\to y} \mu_{\alpha\beta}(x)\mu_{\beta\gamma}(y) \sim \frac{\delta_{\alpha\gamma}}{|x-y|^{2X_\mu(q)}} + (1-\delta_{\alpha\gamma})\frac{C_\mu(q)\ \mu_{\alpha\gamma}(y)}{|x-y|^{X_\mu(q)}} + \dots,
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$$

- Clearly to find C_μ an analytic continuation of some minimal model structure constants is required.

A conjecture for C_μ using Al. Zamolodchikov conformal bootstrap

- **Trick:** To reproduce the three-point function we can just consider **two** operators μ and its charge conjugate $\bar{\mu}$ with S_q invariant OPE

$$\mu \cdot \bar{\mu} \sim I + \dots$$
$$\mu \cdot \mu + \bar{\mu} \cdot \bar{\mu} \sim C_\mu (\mu + \bar{\mu}) + \dots$$

- The symmetric combination $\phi = \frac{\mu + \bar{\mu}}{\sqrt{2}}$, satisfies

$$\phi \cdot \phi \sim I + \frac{C_\mu}{\sqrt{2}} \phi + \dots$$

- ϕ is normalized to one in the identity channel and has multiplicity **one**; since these are the hp under which Al. Zamolodchikov (hep-th/0505063) rederived the structure constants for the A series of minimal models we conjectured

$$\boxed{C_\mu(q) = \sqrt{2} C_{X_\phi X_\phi X_\phi}(c)} \quad c(q) \text{ is the CFT central charge}$$

A numerical check of the conjecture

(R. Ziff, J.Simmons and P.Kleban JPA 2011)

- Random percolation corresponds to $c = 0$ and $X_\phi = X_\mu = \frac{5}{96}$ giving
$$\lim_{q \to 1} C_\mu(q) = 1.022\ldots$$
- A result which agrees with high-precision numerical simulations by Ziff et al.

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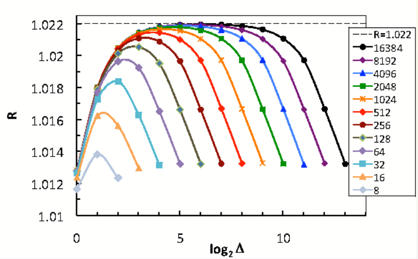

Figure: The universal ratio $C_\mu(q = 1)$, called R, obtained computing the probability of three points on an equilateral triangle of side Δ to be connected. The lattice has p.b.c and size $L \times L$.

Applications to 2d percolation, the cluster size ratio

(G. Delfino, JV and J.Cardy, JPA 2010)

- Again from the FK mapping

$$\frac{\text{cluster size below } p_c}{\text{cluster size above } p_c} = \lim_{q\to 1} \frac{\int d^2x \langle \sigma_\alpha(x)\sigma_\alpha(0)\rangle_{T>T_c}}{\int d^2x \langle \sigma_\alpha(x)\sigma_\alpha(0)\rangle_{T<T_c}}$$

- $\sigma_\alpha(x)$ is the Potts spin field and, in the scaling limit, Potts field theory is **perturbed** away from its critical point

$$\mathcal{A}^{Potts}_{Scaling} = \mathcal{A}^{Potts}_{CFT} + \frac{T - T_c}{T_c} \int d^2x \overbrace{\varepsilon(x)}^{\text{therm. def.}}$$

- The resulting QFT is S_q invariant, massive and **integrable** (L. Chim and A.B. Zamolodchikov, 1992).
- Once the S matrix is known, correlation functions are usually computed as spectral series through the **form factor** axioms.

The end of a long history...

- This is precisely what we did it (although it is **NOT** a standard form factor computation) finding

$$\frac{\Gamma^+}{\Gamma^-} \equiv \frac{\text{cluster size below } p_c}{\text{cluster size above } p_c} \sim 160.2$$

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- Numerical determinations (from a talk of R. Ziff)

year	author	system, method	Γ^+/Γ^-
1976	Skyes, Gaunt, Glen	lattice, series (12-20 order)	1.3-2.0
1976	Stauffer	lattice, series analysis	$\sim$ 100
1978	Nakanishi, Stanley	lattice, MC	25
1978	Wolff, Stauffer	lattice, series	180(36)
1979	Hoshen et al.	lattice, MC	196(40)
1980	Nakanishi, Stanley	lattice, MC(reanalyze)	219(25)
1981	Gawlinsky, Stanley	overlapping disk, MC	50(26)
1985	Rushton, Familiy, Herrmann	additive polymerization, MC	140(45)
1987	Kim, Herrmann, Landau	continuum model, MC	14(10)
1987	Nakanishi	AB percolation, MC	139(24)
1988	Balberg	widthless sticks, MC	$\sim$ 3
1989	Corsten, Jan, Jerrard	lattice, MC	75(+40, -25)
1990	S.B. Lee, Torquato	penetrable conc. shell	1050(32)
1990	S.B. Lee	disks, MC	192(20)
1993	Zhang, De'Bell	Penrose quasi-lattice, series	310(60)
1995	Conway, Guttman	lattice, series (26-33 order)	45(+20, -10)
1996	S.B. Lee	penetrable conc. shell, disks	175(50)
1997	S.B. Lee, Jeon	kinetic gelation, MC	170(20)
2006	Jensen, Ziff	lattice, MC, series	162.5 (2.0)

A short summary of this talk

- We predicted the exact formula for the three-point connectivity in two-dimensional percolation at criticality.
- We gave an accurate estimation for the off-critical cluster size ratio and we also computed all the independent universal ratios in two-dimensional percolation.
- Our results which rely on techniques of Integrable and Conformal Field Theory, agree with the best numerical simulations at disposal.
- Thank you for your attention!