Non-Linear waves on the Edge of FQHE:

Quantum Benjamin-Ono Equation - case study of quantum hydrodynamics

P. Wiegmann

(Joint work with friends: Abanov, et. al.)

Bologna

September 20, 2011

Messages

- about FQH Edge states
- about Benjamin-Ono equation
- Relation between Benjamin-Ono equation and CFT
- Relation between FQHE and CFT

FQHE -Laughlin's state(s)

Particles on a plane in a quantized magnetic field (with a strong Coulomb Interaction)

$$\Psi_0(z_1,\ldots,z_N) = \left[\Delta(z_1,\ldots,z_N)\right]^{\beta} e^{-\frac{1}{2}\sum_i |z_i|^2/4\ell^2}$$

- $\Delta = \prod_{i\neq j}(z_i - z_j)$- VanDerMonde determinant
- ℓ -magnetic length;
- $\nu = 1/\beta$ - is a filling fraction;
- $\beta = 1$ - IQHE; $\beta = 3$- FQHE.

Important features:

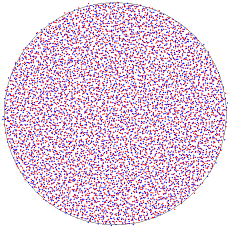
- Wave-function is holomorphic;
- Degree of zero at $z_i \to z_j$ is larger than 1;

Gauge and a shape of a droplet

- The w.f. $\Delta^\beta e^{-\frac{1}{2}\sum_i |z_i|^2/4\ell^2}$ is written in a radial gauge.
- A gauge transformation

$$\Psi \to e^{\frac{1}{\ell^2}\sum_i V(z_i)}\Psi, \quad V(z) \quad \text{is analytic inside the droplet}$$

does not change magnetic field but change a shape of the droplet.



- The ground state is $\nu \times$ Volume degenerate.
- Potential well lifts degeneracy

Edge states

- Degenerate Ground state $H_0 \Psi_V = 0$,
- Spectrum separated from the ground state by a large gap ℓ^{-1}
- Potential well lifts a degeneracy:

$$H_0 \to H = H_0 + \sum_i A_0(|r_i|)$$

- Evolution within almost (infinitely) degenerate states

$$\Psi_V(t) = e^{-iHt}\Psi_V$$

$$\nabla A_0 \ell \ll 1; \quad k\ell \ll 1$$

- Low energy states are localized on an edge $\to$ Edge States

Linear Edge States theory

Accepted theory of the Edge state

- Density is a chiral field:

$$\rho(x) = \sum_k e^{ikx} \rho_k, \quad \rho_k = \rho^\dagger_{-k},$$

$$[\rho_k, \rho_l] = \nu k \delta_{k+l},$$

$$(\partial_t - v_0 \nabla)\rho = 0, \quad v_0 = \frac{\hbar}{m} A_0'.$$

- Common believe (I disagree with):

  $c = 1$ -CFT of free bosons with a compactification radius $\nu = \beta^{-1}$.

Problem of Edge states

Data:

- Analyticity;
- Degree of zeros β is larger than 1;
- Degeneracy;
- Separation of scales

Goal: Universal dynamics on the edge.

Model Hamiltonian

For concreteness

- $D_i = \partial_i - \sum_{j\neq i} \frac{\beta}{z_i - z_j} - A(z_i);, \quad \nabla \times A = B$
- $D_i \Psi_0 = 0$
- Model Hamiltonian

$$H = \sum_i \frac{\hbar^2}{2m} D_i^\dagger D_i + A_0(|z_i|), \quad \frac{\hbar^2}{2m\ell^2} \gg A_0.$$

- Being projected onto the L-space

$$H = \sum_i \frac{\hbar^2}{2m\ell^2} |\pi(z_i) + V'(z_i)|^2 + A_0(|z_i|)$$

$$[\pi(z_i), V(z_j)] = i\beta \log(z_i - z_j)$$

Problem of Edge States

- Given space of states (L-states)

$$\Psi_V = \Delta^\beta e^{-\frac{1}{4\ell^2}\sum_i\left(|z_i|^2+V(z_i)\right)}$$

- Given the Hamiltonian

$$H = \sum_i \frac{\hbar^2}{2m\ell^2}|\pi(z_i)+V'(z_i)|^2 + A_0(|z_i|), \quad [\pi(z_i), V(z_j)] = i\beta\log(z_i - z_j)$$

- Compute dynamics of density and velocity

$$v(z) = i\pi(z) + V'(z), \quad [\rho(z), v(z')] = i\beta\nabla\delta(z-z')$$

- Project on the L-space:

$$m\hbar^{-2}\ell^2|\nabla A_0| \to 0$$

- Integrate over the bulk, isolate boundary states;

Non-linear theory of Edge States

- Linearized version: $(\partial_t - v_0 \nabla)\rho = 0, \quad v_0 = \frac{\hbar}{m} A_0'$
- Universal description of non-linear chiral boson at FQHE edge (in units $\ell = 1\ \hbar/m = 1$)

$$\boxed{(\partial_t - v_0 \nabla)\rho + \nabla\left[\rho\left(\rho + \frac{1}{4}(1-\beta)\nabla\left(\log\rho\right)_H\right)\right] = 0}$$

$$[\rho(x), \rho(x')] = \nu \nabla \delta(x - x'),$$

$$f_H(x) = \frac{1}{\pi} P.V. \int \frac{f(x')}{x - x'} dx'$$

Fundamental chiral equation $\longrightarrow$ Quantum Benjamin-Ono Equation

- Expansion around mean density (or Backlund transformation)

$$\rho \approx \rho_0 + \nabla \varphi, \quad \dot{\rho} + \nabla \left[\rho \left(\rho + \frac{1}{4}(1-\beta)\nabla \left(\log \rho\right)_H \right) \right] = 0$$

- Quantum Benjamin-Ono Equation

$$\boxed{\dot{\varphi} + \tfrac{1}{2}\left(\nabla \varphi\right)^2 + \tfrac{1}{4}(1-\beta)\nabla^2 \varphi_H = 0}$$

$$[\varphi(x),\, \varphi_H(x')] = \nu \,\mathrm{Im} \log(x - x' + i0),$$

$$\varphi_H(x) = \frac{1}{\pi} P.V. \int \frac{\varphi(x')}{x - x'} dx'$$

Origin of Benjamin-Ono and Fundamental Chiral Equations

$$\boxed{\dot{\varphi} + \frac{1}{2}\left(\nabla\varphi\right)^2 + \frac{1}{4}(1-\beta)\nabla^2\varphi_H = 0}$$

- Classical Benjamin-Ono Equation
  Surface waves of interface of stratified fluids: Incompressible fluids with a sharp change of density (*Benjamin 1968*).

- Quantum Benjamin-Ono Equation and Fundamental Chiral Equation

$$\boxed{\dot{\rho} + \nabla\left[\rho\left(\rho + \frac{1}{4}(1-\beta)\nabla\left(\log\rho\right)_H\right)\right] = 0}$$

  A chiral Sector of Calogero -Sutherland model (*Abanov,Betelheim,P.W.,2006*)

$$H = \sum_i \frac{1}{2}\partial_{x_i}^2 + \sum_{i>j} \frac{\beta(\beta-1)}{\sin^2(x_i - x_j)}$$

- Completes a prove of a long-standing conjecture by Sakita (?) that Edge State of FQHE= Calogero model.

LIMITING CASES: ILW *-Intermediate Long-Wave Equation*

$$\boxed{\dot{\varphi} + \tfrac{1}{2}\left(\nabla\varphi\right)^2 + \tfrac{1}{4}(1-\beta)\nabla^2\varphi_H = 0},$$

$$\varphi_H(x) = \frac{1}{\pi L} P.V. \int \cot\left(\frac{x-x'}{L}\right)\varphi(x')dx'$$

- Deep sea $\boxed{L \to \infty : \text{ILW} \to \text{Benjamin-Ono Equation}}$
- Shallow lake $\boxed{L \to 0 : \text{ILW} \to \text{KdV}}$

$$\dot{\varphi} + \frac{1}{2}\left(\nabla\varphi\right)^2 + \nabla^3\varphi = 0$$

- Quantum ILW: A chiral Sector of hyperbolic Calogero -Sutherland model *(P.W.,2006)*

$$H = \sum_i \frac{1}{2}\partial_{x_i}^2 + \sum_{i>j} \frac{\beta(\beta-1)}{L\sinh^2\frac{(x_i-x_j)}{L}}$$

INTEGRABILITY

$$\boxed{\dot{\varphi} + \frac{1}{2}\left(\nabla\varphi\right)^2 + \frac{1}{4}(1-\beta)\nabla^2\varphi_H = 0}$$

- ILW and Benjamin-Ono Equations (*Krichever, 1971, Satsuma 1978*)
- ILW as a real reduction of MKP hierarchy (P.W., 2006)
- Fundamental Chiral Equation and Benjamin-Ono Eq as a Backlund transformation of Chiral Equation (*Abanov, Bettelheim, P.W. , 2019*)
- Integrals can be explicitly constructed.

- Classical Benjamin-Ono equation for an arbitrary domain

$$D \to \mathbb{D} : w = f(z)$$

$$\frac{\partial}{\partial t} \log |f'| = \frac{c}{12} \mathrm{Re}\{f, z\}$$

Properties

$$\boxed{\dot{\varphi} + \frac{1}{2}\left(\nabla\varphi\right)^2 + \frac{1}{4}(1-\beta)\nabla^2\varphi_H = 0}$$

- Two branches of solitons:
  - subsonic: *holes* propagating to the left;

    Charge $\nu = 1/\beta$: $\int \rho_h dx = \text{integer} \times \nu$
  - ultrasonic: *particles* propagating to the right:

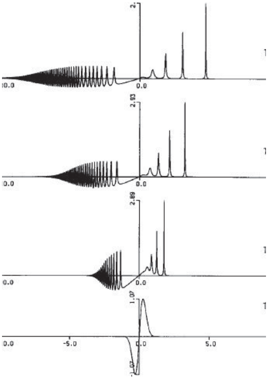
    Charge 1: $\int \rho_p dx = \text{integer}$

$$\rho \equiv -\nabla\varphi = \frac{1}{\pi}\frac{v}{v^2(x-vt)^2+v^2}$$

- Classical Benjamin-Ono equation has only one branch - *particles*

$$\boxed{\text{Benjamin-Ono is the only integrable equation with a quantized charge of solitons}}$$

Two branches of excitations



- Separation between holes (moving right) and particles (moving left)

Benjamin-Ono Equation as a Deformed Boundary CFT

- Boundary CFT occurs outside of the droplet;
- Boundary stress energy tensor components connected by the Cauchy-Riemann condition

$$T_{nn} = -T_{ss} = \mathrm{Re}T$$

- Components:

$$T_{nn} = \frac{1}{2}(\partial_n \varphi)^2 + \frac{\beta - 1}{4}\partial_n \partial_s \varphi$$

- Benjamin-Ono Equation is a deformation of CFT:

$$\boxed{\dot{\varphi} = T_{nn}}$$

- Hydrodynamics form (conservation laws): $\rho = \nabla \varphi$

$$\boxed{\dot{\rho} = \nabla T_{nn}}$$

- Deformation of a boundary is generated by the normal components of the stress-energy tensor.

CFT AND FQHE

- FQHE Edge hydrodynamics is a deformation of **Boundary** CFT with $c = 1 - 6\left(\sqrt{\beta} - 1/\sqrt{\beta}\right)^2 < 1$

- CFT lives outside of a droplet;

Contrary to a common believe that FQHE $\quad c = 1$ bulk CFT.

Subtleties and main steps

Quantum Hydrodynamics in the bulk

- Laughlin's state reformulated as a hydrodynamics if the bulk:
- Density and Velocity

$$[v(r),\rho(r')] = -i\nabla\delta(r-r'); \qquad \text{canonical hydro-variables,}$$
$$\dot{\rho} + \nabla(\rho v) = 0, \qquad \text{continuity equations}$$

- Projection on holomorphic states

$$\nabla\cdot v = 0; \qquad \text{incompressibility,}$$
$$v = \nabla\times\Psi, \qquad \text{stream function,}$$
$$\int(\rho v)\times dr = \beta^{-1}A_0, \qquad \text{Hall Effect}$$

- Degree of zeros

$$[v(r)\times v(r')] = 2\pi\beta i\delta(r-r'), \qquad \text{Heisenberg algebra}$$
$$[\Psi(r),\, \Psi(r')] = \beta\log|r-r'|, \qquad \text{Stream function is a Gaussian Field}$$

Subtleties

- Short-distance anomaly or OPE

$$\rho v =: \rho v : -\frac{\beta}{4}\nabla^*\langle\rho\rangle$$

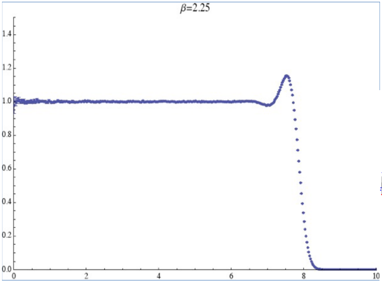
- Velocity of *orbits* vs velocity of particles:

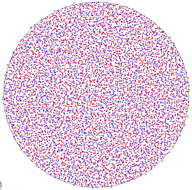
$$\rho v \to \rho v - \frac{2-\beta}{4}\nabla^*\rho$$

- Dipole moment and singularity on the boundary

$$d_0 = \int_0^R (r-R)\rho(r)dr = \frac{\beta}{2\pi}\frac{2-\beta}{4\beta}$$

OVERSHOOT: DIPOLE MOMENT OF FQHE DROPLET

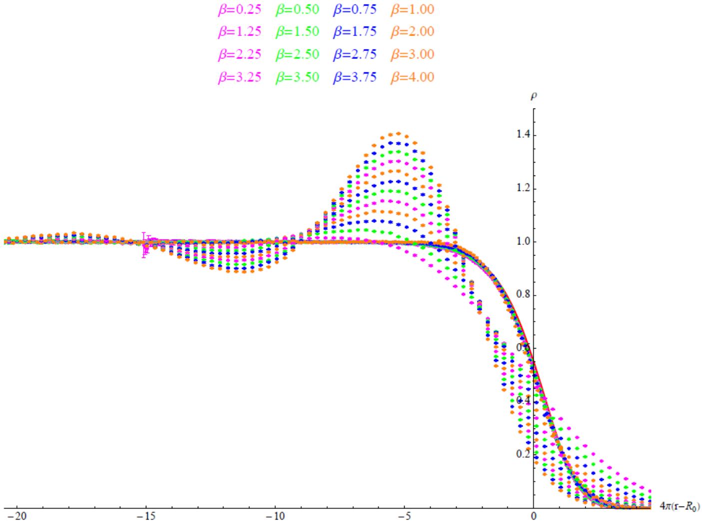




$$\Psi_0 = \left[\prod_{i>j}(z_i - z_j)\right]^{\beta} e^{-\frac{1}{2}\sum_i |z_i|^2/4\ell^2},$$

$$d_0 = \int_0^R (r - R)|\rho(r)dr = \frac{\beta}{2\pi}\frac{2-\beta}{4\beta}$$

Dipole moment of a spherical droplet



$$\langle \rho \rangle = \lim_{N\to\infty} \int |z - z_i|^{2\beta} |\Delta|^{2\beta} e^{-\sum_i |z_i|^2} \prod_i d^2 z_i$$

No overshoot at $\beta \leq 1$.

Summary

- Boundary waves in FQHE are essentially nonlinear;
- They are generated by a stress energy tensor of CFT with $c = 1 - 6\nu^{-1}(\nu - 1)^2 < 1$ situated outside of the droplet
- An origin of shifting the central charge is a dipole moment located on the boundary of the droplet

Density along different axis of elliptical droplet;