Fisica Generale A

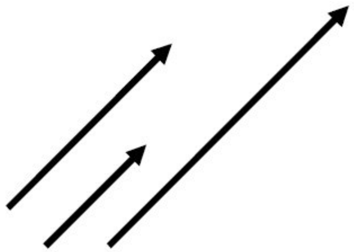
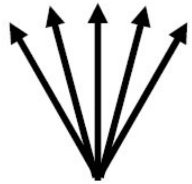
Vettori

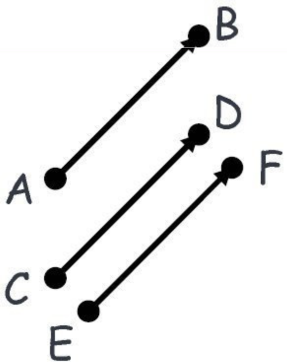
Scuola di Ingegneria e Architettura

UNIBO – Cesena

Anno Accademico 2015 – 2016

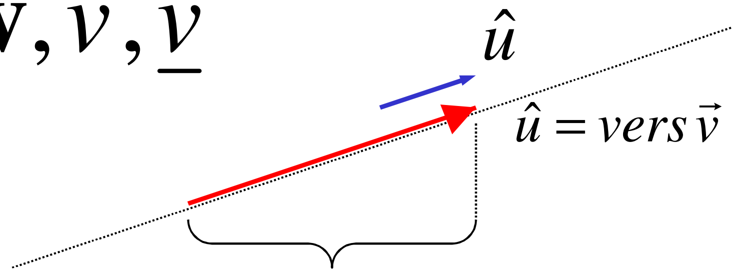
Vettori

$$\vec{v} = B - A = D - C = F - E$$

$$\vec{v},\ \mathbf{v},\ \bar{v},\ \underline{v}$$

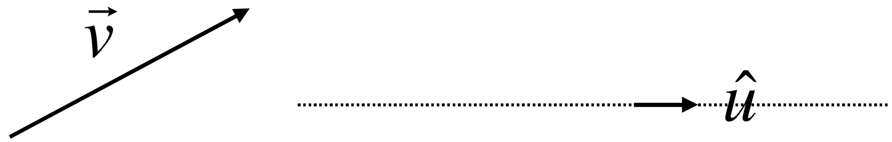


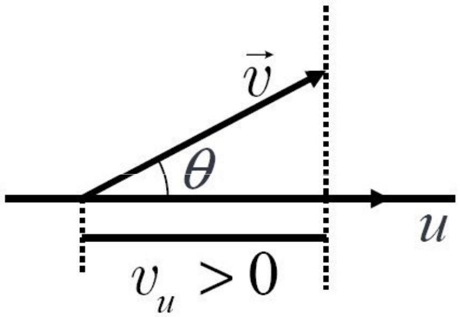
$$\hat{u} = vers\,\vec{v} = \hat{v}$$

$$\left\|\vec{v}\right\| = \left|\vec{v}\right| = v$$

$$\vec{v} = \left\|\vec{v}\right\|\hat{u} = \left|\vec{v}\right|\hat{u} = v\hat{u}$$

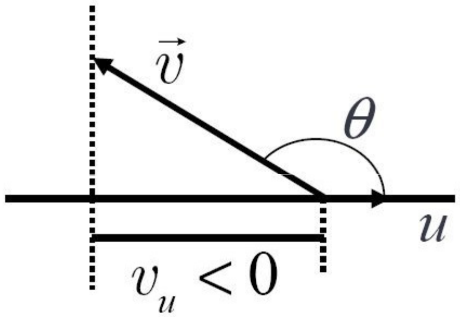
Vettori

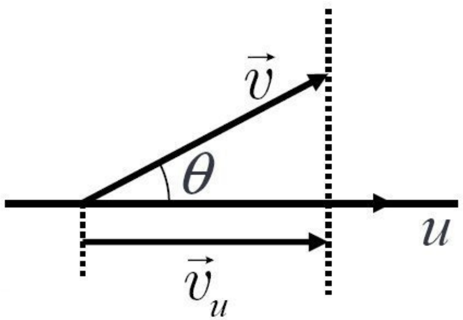




$$v_u = v \cos\theta$$

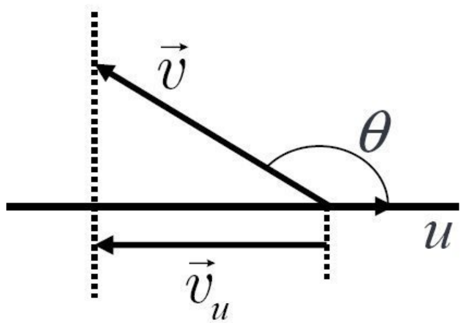
la componente



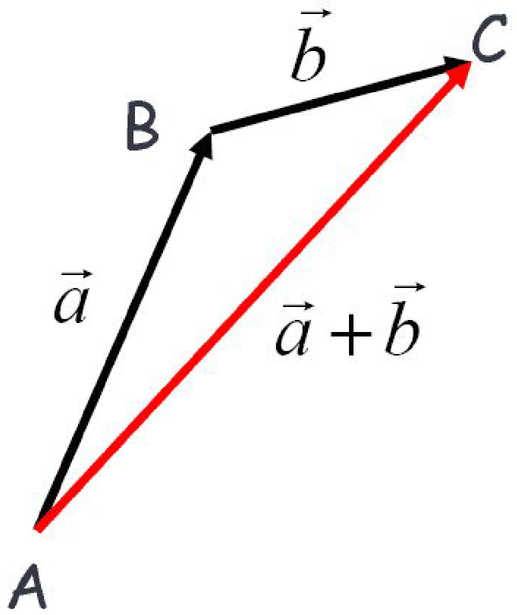


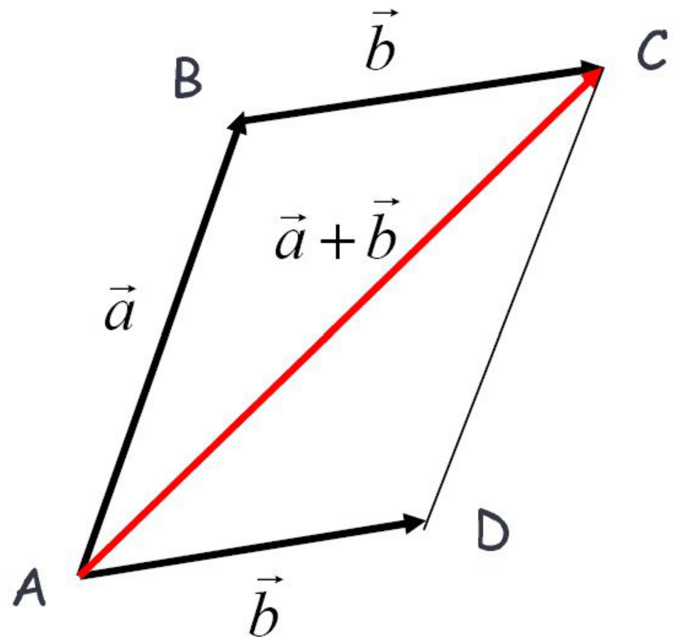
$$\vec{v}_u = v \cos\theta \, \hat{u}$$

il componente



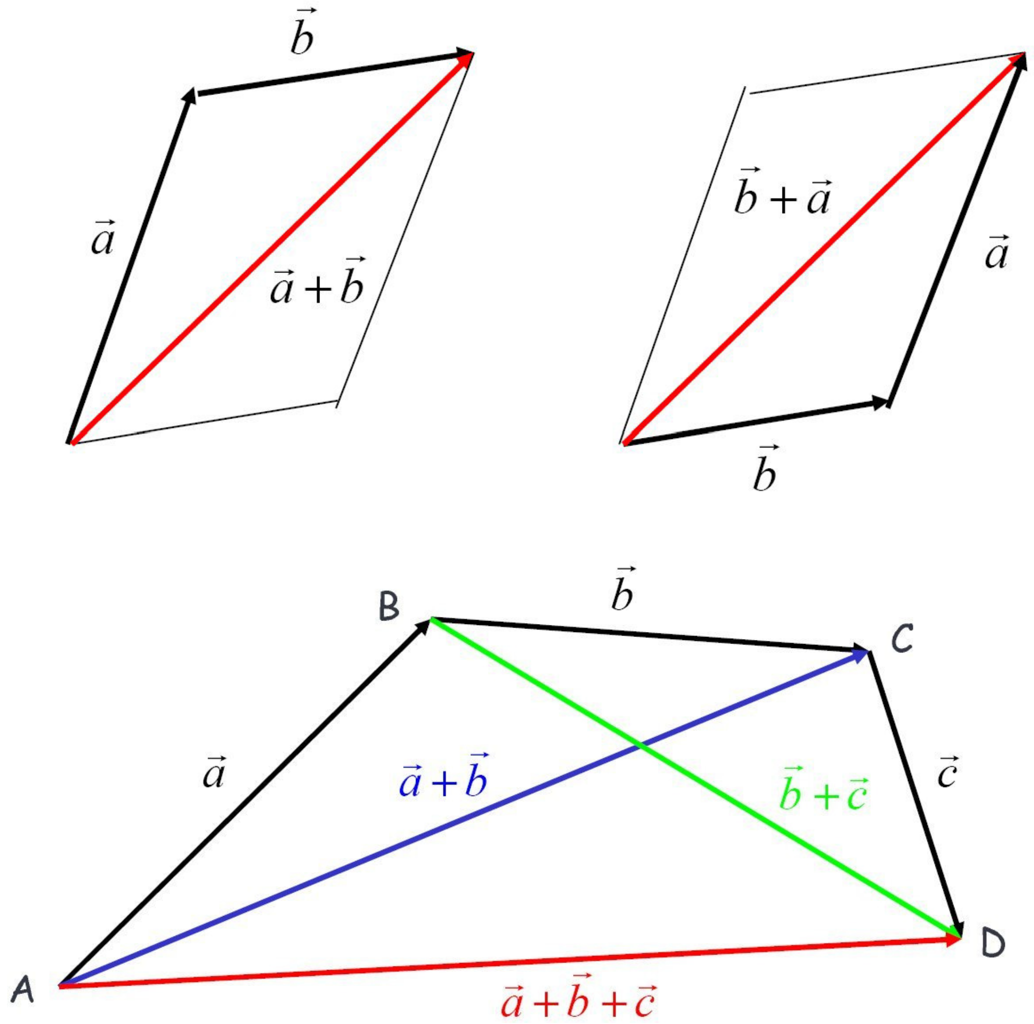
Vettori - somma



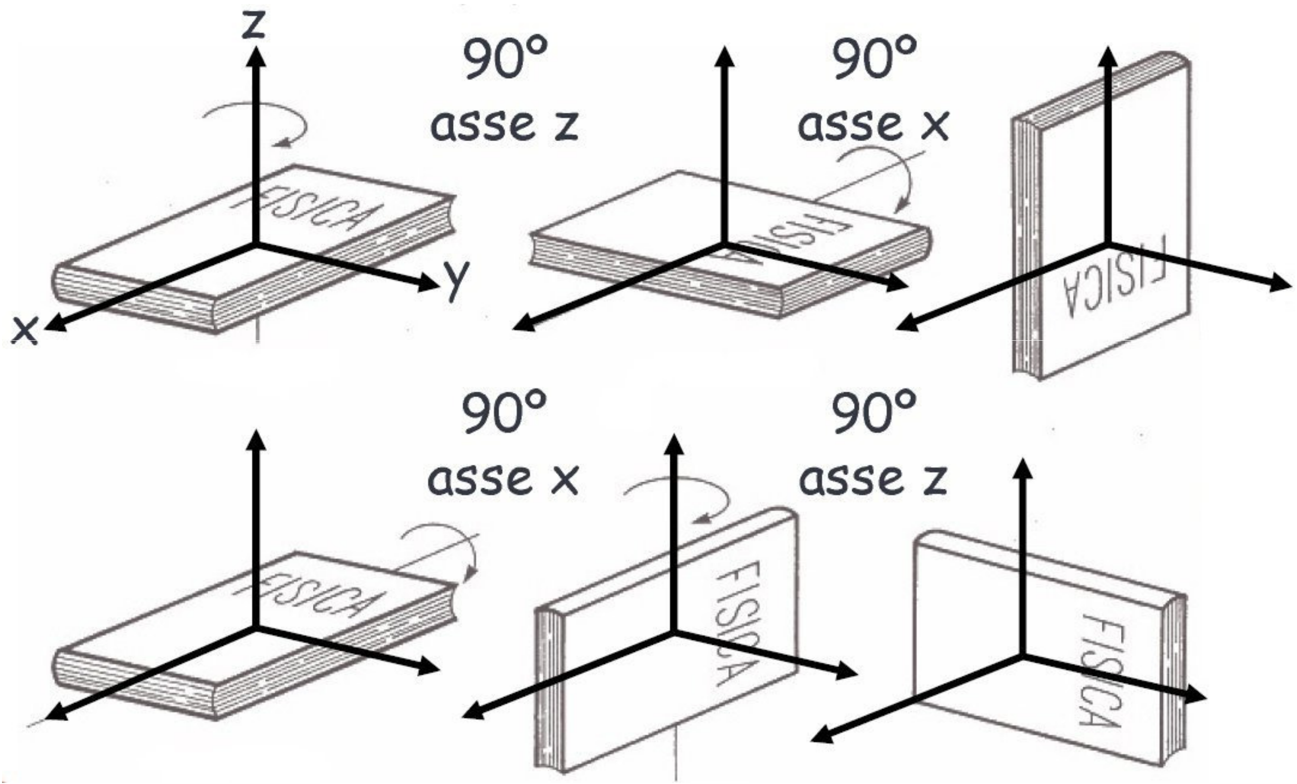


$$(B-A)+(C-B)=(C-A)$$

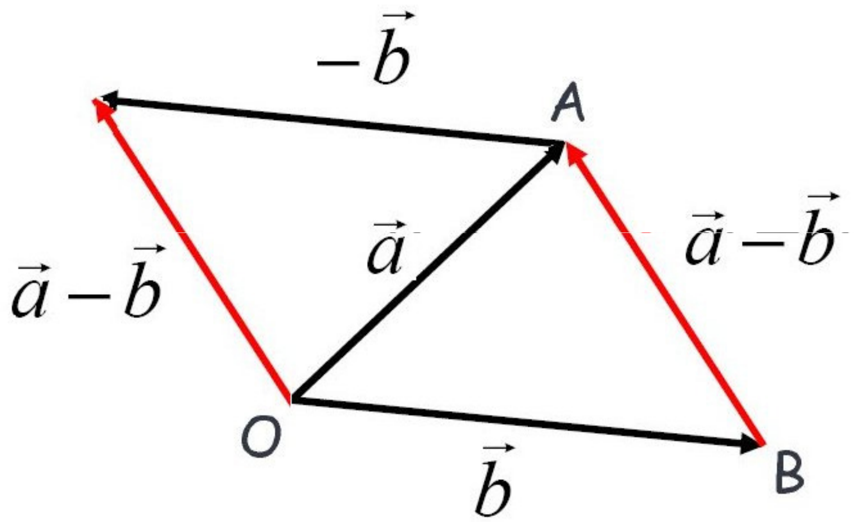
Vettori - somma



Vettori



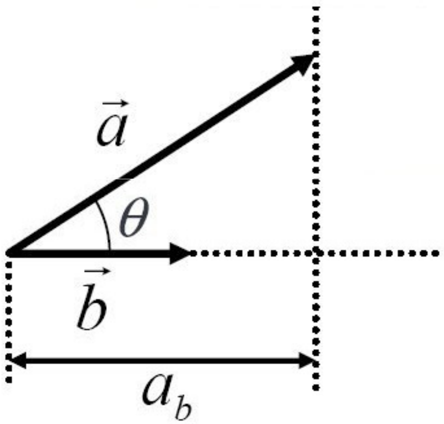
Vettori - differenza

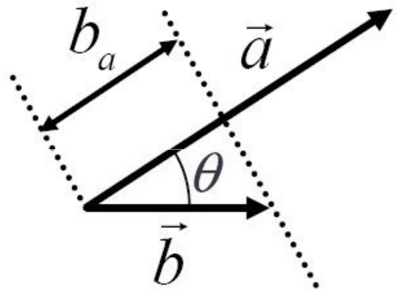


$$(A-O)-(B-O)=(A-B)$$

Vettori – prodotto scalare

$$\vec{a} \cdot \vec{b} = ab \cos \vartheta$$





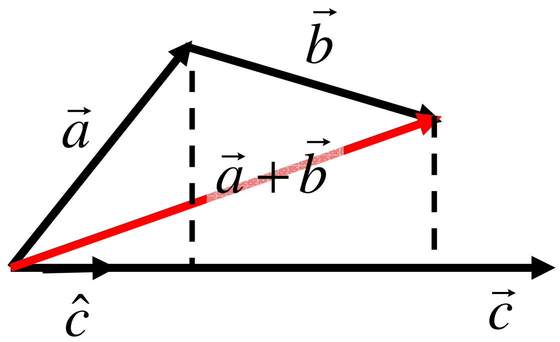
$$\vec{a} \cdot \vec{b} = a_b b$$

$$\vec{a} \cdot \vec{b} = a b_a$$

$$\vec{a} \cdot \vec{a} = \vec{a}^2 = a^2 = |\vec{a}|^2 = \|\vec{a}\|^2$$

Vettori – prodotto scalare

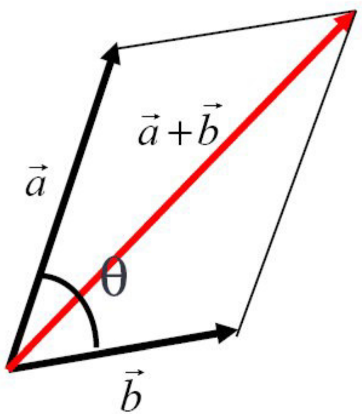
Proprietà distributiva



$$\vec{c}\cdot\left(\vec{a}+\vec{b}\right)=c\left(\vec{a}+\vec{b}\right)\cdot\hat{c}\;=c\left(\vec{a}+\vec{b}\right)_c$$
$$=c\left(a_c+b_c\right)\;=ca_c+cb_c$$
$$=\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}$$

$$\boxed{\vec{c}\cdot\left(\vec{a}+\vec{b}\right)=\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}}$$

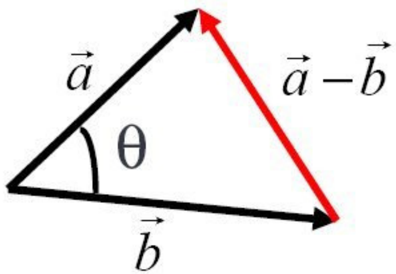
Vettori



$$\left\|\vec{a}+\vec{b}\right\| = \sqrt{\left(\vec{a}+\vec{b}\right)^2} = \sqrt{\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right)} =$$

$$= \sqrt{\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}}$$

$$\left\|\vec{a}+\vec{b}\right\| = \sqrt{a^2+b^2+2ab\cos\vartheta}$$



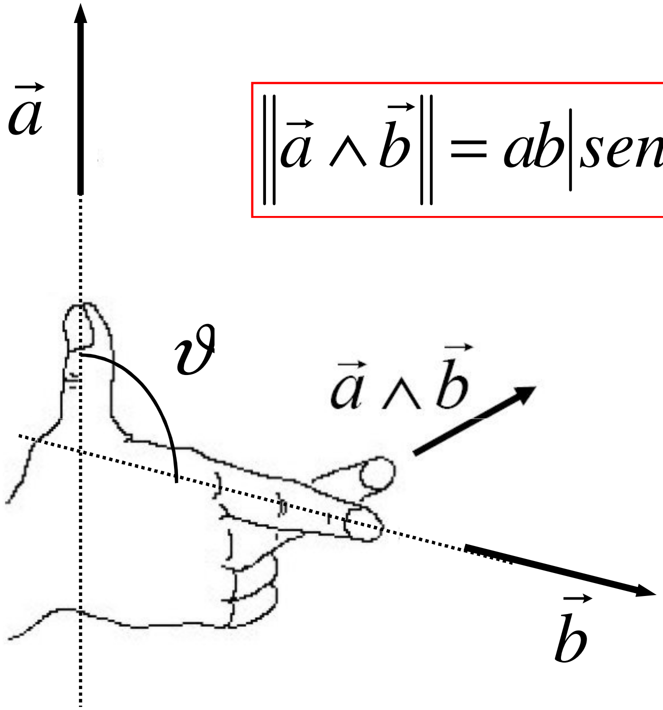
$$\left\|\vec{a}-\vec{b}\right\| = \sqrt{\left(\vec{a}-\vec{b}\right)^2} = \sqrt{\left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)} =$$

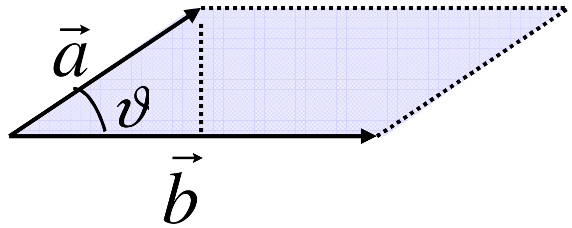
$$= \sqrt{\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}}$$

$$\left\|\vec{a}-\vec{b}\right\| = \sqrt{a^2+b^2-2ab\cos\vartheta}$$

Vettori – prodotto vettoriale

$$\left\|\vec{a} \wedge \vec{b}\right\| = ab\left|sen\,\vartheta\right|$$





Regola della mano destra.

Proprietà:

$$\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a}$$

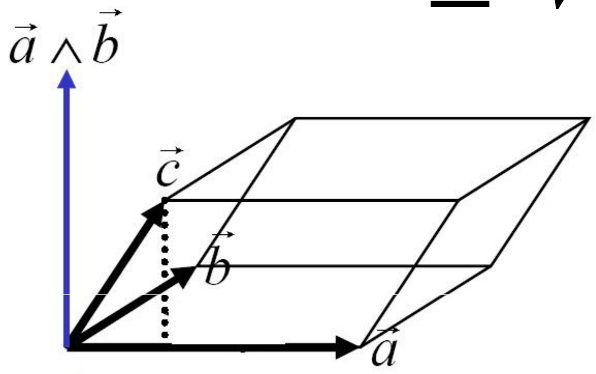
$$\left(\vec{a} \wedge \vec{b}\right) \wedge \vec{c} \neq \vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right)$$

Esempio: *siano* $\hat{a}$ *e* $\hat{b}$ *due versori ortogonali* $\Rightarrow$

$$\left(\hat{a} \wedge \hat{a}\right) \wedge \hat{b} = \vec{0}\,; \quad \hat{a} \wedge (\hat{a} \wedge \hat{b}) = -\hat{b}$$

Vettori – doppio prodotto misto

$$\pm V = \vec{a} \wedge \vec{b} \cdot \vec{c}$$



$$B = \left\| \vec{a} \wedge \vec{b} \right\|$$

$$h = \left| \vec{c} \cdot vers\left(\vec{a} \wedge \vec{b}\right) \right|$$

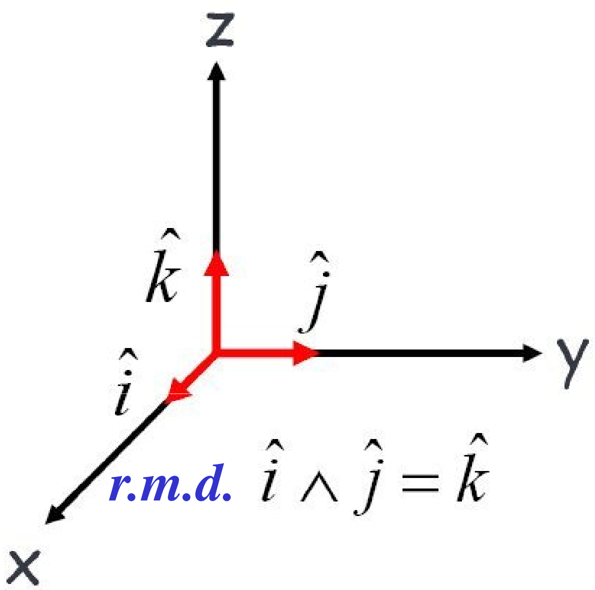
$$V = Bh = \left| \left\| \vec{a} \wedge \vec{b} \right\| vers\left(\vec{a} \wedge \vec{b}\right) \cdot \vec{c} \right| = \left| \vec{a} \wedge \vec{b} \cdot \vec{c} \right|$$

Proprietà:

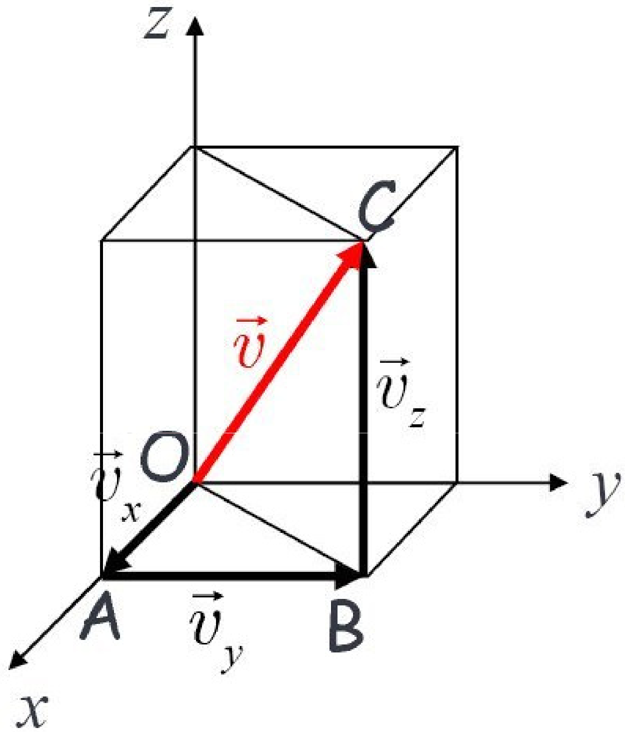
$$\vec{a} \wedge \vec{b} \cdot \vec{c} = \vec{b} \wedge \vec{c} \cdot \vec{a} = \vec{c} \wedge \vec{a} \cdot \vec{b}$$

$$\vec{a} \wedge \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \wedge \vec{c}$$

Vettori – rappresentazione cartesiana



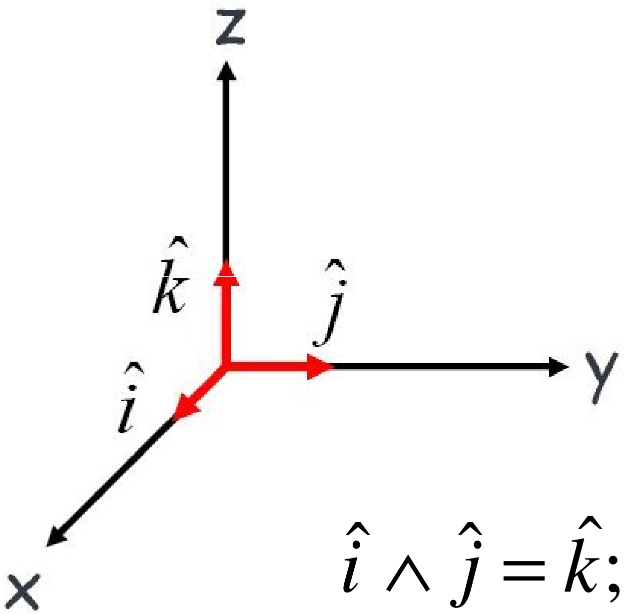
r.m.d. $\hat{i} \wedge \hat{j} = \hat{k}$



$$\vec{v} = \begin{cases} \vec{v}_x + \vec{v}_y + \vec{v}_z \\ v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \end{cases}$$

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

Vettori – rappresentazione cartesiana



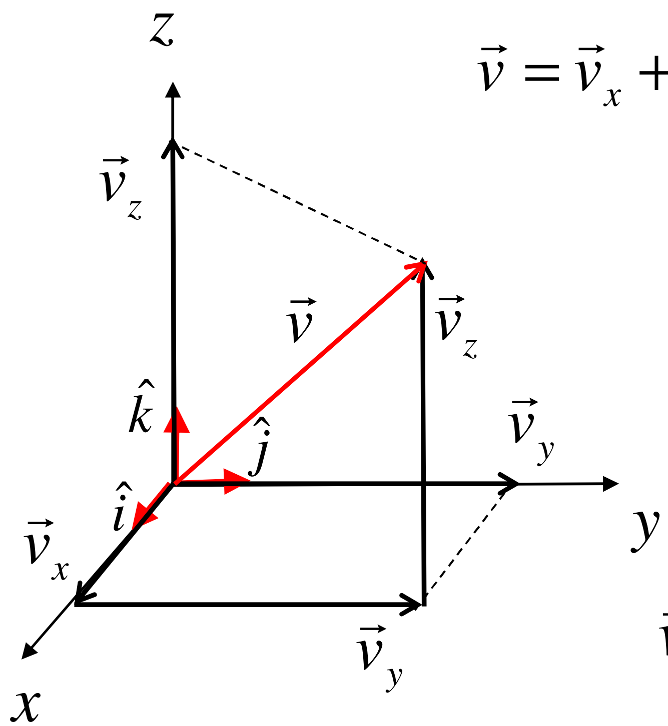
$$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$$

$$\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$$

$$\hat{i}\wedge\hat{i}=\hat{j}\wedge\hat{j}=\hat{k}\wedge\hat{k}=0$$

$$\hat{i}\wedge\hat{j}=\hat{k};\quad \hat{j}\wedge\hat{k}=\hat{i};\quad \hat{k}\wedge\hat{i}=\hat{j}$$

Vettori – rappresentazione cartesiana



$$\vec{v} = \vec{v}_x + \vec{v}_y + \vec{v}_z = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\vec{v}\cdot\hat{i} = v_x = v\cos\alpha$$

$$\vec{v}\cdot\hat{j} = v_y = v\cos\beta$$

$$\vec{v}\cdot\hat{k} = v_z = v\cos\gamma$$

→ *Coseni direttori*

$$\vec{v} = v\left(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}\right)$$

$$v^2 = v_x^2 + v_y^2 + v_z^2 = v^2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)$$

⇒ $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Vettori – operazioni nella r. c.

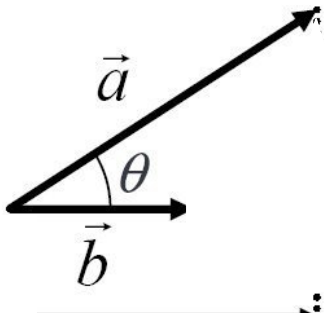
$$\vec{a} + \vec{b} = \vec{c}$$

$$\vec{a} + \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) + (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) =$$

$$= (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

$$c_s = a_s + b_s \quad s = x, y, z$$

Vettori – operazioni nella r. c.



$$\vec{a}\cdot\vec{b}=c$$

$$\vec{a}\cdot\vec{b}=(a_x\hat{i}+a_y\hat{j}+a_z\hat{k})\cdot(b_x\hat{i}+b_y\hat{j}+b_z\hat{k})=$$

$$=a_xb_x\hat{i}\cdot\hat{i}+a_x\cancel{b_y\hat{i}\cdot\hat{j}}+a_x\cancel{b_z\hat{i}\cdot\hat{k}}+a_y\cancel{b_x\hat{j}\cdot\hat{i}}+a_yb_y\hat{j}\cdot\hat{j}+a_y\cancel{b_z\hat{j}\cdot\hat{k}}+$$

$$+a_z\cancel{b_x\hat{k}\cdot\hat{i}}+a_z\cancel{b_y\hat{k}\cdot\hat{j}}+a_zb_z\hat{k}\cdot\hat{k}\;=a_xb_x+a_yb_y+a_zb_z=c$$

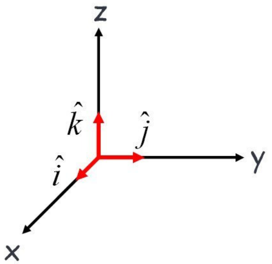
$$\vec{a}\cdot\vec{b}=a\left(\cos\alpha_a\,\hat{i}+\cos\beta_a\,\hat{j}+\cos\gamma_a\,\hat{k}\right)\cdot b\left(\cos\alpha_b\,\hat{i}+\cos\beta_b\,\hat{j}+\cos\gamma_b\,\hat{k}\right)$$

$$=ab\left(\cos\alpha_a\cos\alpha_b+\cos\beta_a\cos\beta_b+\cos\gamma_a\cos\gamma_b\right)$$

$$=ab\cos\theta$$

$$\cos\theta=\left(\cos\alpha_a\cos\alpha_b+\cos\beta_a\cos\beta_b+\cos\gamma_a\cos\gamma_b\right)$$

Vettori – operazioni nella r. c.



$$\vec{a} \wedge \vec{b} = \vec{c}$$

$$\vec{a} \wedge \vec{b} = \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) \wedge \left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right) =$$

$$= a_x b_x \underbrace{\hat{i} \wedge \hat{i}}_{0} + a_x b_y \underbrace{\hat{i} \wedge \hat{j}}_{\hat{k}} + a_x b_z \underbrace{\hat{i} \wedge \hat{k}}_{-\hat{j}} + a_y b_x \underbrace{\hat{j} \wedge \hat{i}}_{-\hat{k}} + a_y b_y \underbrace{\hat{j} \wedge \hat{j}}_{0} + a_y b_z \underbrace{\hat{j} \wedge \hat{k}}_{\hat{i}} +$$

$$+ a_z b_x \underbrace{\hat{k} \wedge \hat{i}}_{\hat{j}} + a_z b_y \underbrace{\hat{k} \wedge \hat{j}}_{-\hat{i}} + a_z b_z \underbrace{\hat{k} \wedge \hat{k}}_{0} =$$

$$= \left(a_y b_z - a_z b_y\right)\hat{i} + \left(a_z b_x - a_x b_z\right)\hat{j} + \left(a_x b_y - a_y b_x\right)\hat{k} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

$$\vec{a} \wedge \vec{b} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

$$\left(\vec{a} + \vec{b}\right) \wedge \vec{c} = \vec{a} \wedge \vec{c} + \vec{b} \wedge \vec{c} \quad ?$$

Vettori – operazioni nella r. c.

$$\vec{a}\wedge\vec{b}\bullet\vec{c}=\left[\left(a_y b_z-a_z b_y\right)\hat{i}+\left(a_z b_x-a_x b_z\right)\hat{j}+\left(a_x b_y-a_y b_x\right)\hat{k}\right]\bullet\left(c_x\hat{i}+c_y\hat{j}+c_z\hat{k}\right)=$$

$$=\left(a_y b_z-a_z b_y\right)c_x+\left(a_z b_x-a_x b_z\right)c_y+\left(a_x b_y-a_y b_x\right)c_z$$

$$=a_y b_z c_x-a_z b_y c_x+a_z b_x c_y-a_x b_z c_y+a_x b_y c_z-a_y b_x c_z$$

$$=(b_y c_z-b_z c_y)a_x+(b_z c_x-b_x c_z)a_y+(b_x c_y-b_y c_x)a_z=\vec{b}\wedge\vec{c}\cdot\vec{a}$$

$$=(c_y a_z-c_z a_y)b_x+(c_z a_x-c_x a_z)b_y+(c_x a_y-c_y a_x)b_z=\vec{c}\wedge\vec{a}\cdot\vec{b}$$

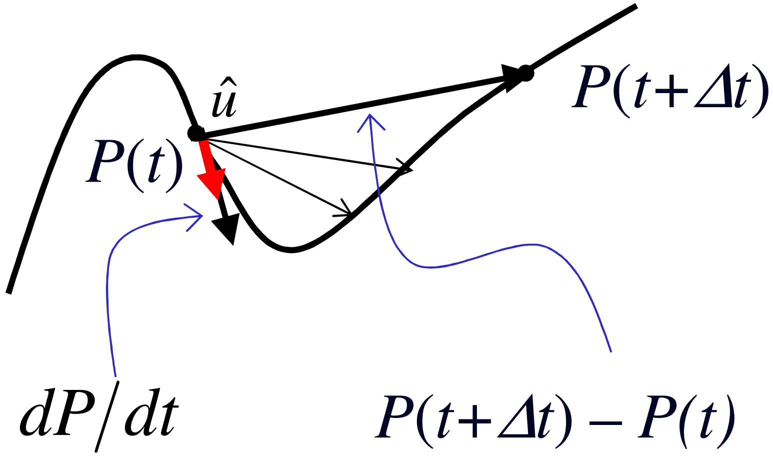
$$\vec{a}\wedge\vec{b}\cdot\vec{c}=\det\begin{pmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{pmatrix}$$

$\frac{dy}{dx}$

Vettori – derivata di un punto

$$P = P(t)$$

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t}$$



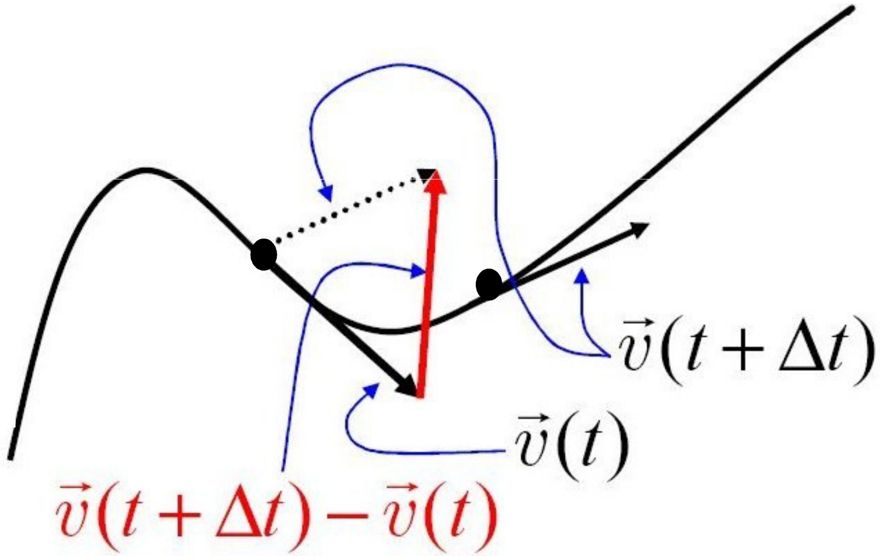
$$\frac{dP}{dt} = \left|\frac{dP}{dt}\right| \hat{u}$$

$\frac{dP}{dt}$ *è un vettore con direzione tangente alla traiettoria nel punto P*

Vettori – derivata di un vettore

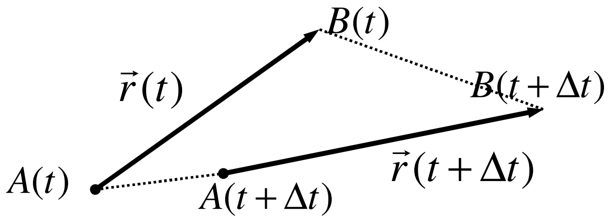
$$\vec{v} = \vec{v}(t)$$

$$\frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$



$\dfrac{d\vec{v}}{dt}$ *è un vettore*

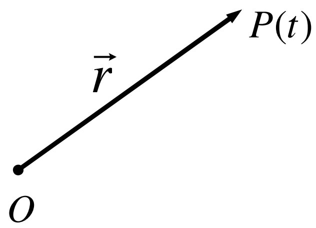
Vettori – derivate



$$\frac{d}{dt}(B-A)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{[B(t+\Delta t)-A(t+\Delta t)]-[B(t)-A(t)]\}$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{[B(t+\Delta t)-B(t)]-[A(t+\Delta t)-A(t)]\}\quad =\lim_{\Delta t\to 0}\frac{B(t+\Delta t)-B(t)}{\Delta t}-\lim_{\Delta t\to 0}\frac{A(t+\Delta t)-A(t)}{\Delta t}=\frac{dB}{dt}-\frac{dA}{dt}$$

Caso particolare



$$\frac{d\vec{r}}{dt}=\frac{d}{dt}[P(t)-O]=\frac{dP}{dt}$$

Regole di derivazione

$$\frac{d}{dt}\left(\vec{a}\pm\vec{b}\right)=\frac{d\vec{a}}{dt}\pm\frac{d\vec{b}}{dt}$$

$$\frac{d}{dt}\left(\alpha\vec{a}\right)=\frac{d\alpha}{dt}\vec{a}+\alpha\frac{d\vec{a}}{dt}$$

$$\frac{d}{dt}\left(\vec{a}\bullet\vec{b}\right)=\frac{d\vec{a}}{dt}\bullet\vec{b}+\vec{a}\bullet\frac{d\vec{b}}{dt}$$

$$\frac{d}{dt}\left(\vec{a}\wedge\vec{b}\right)=\frac{d\vec{a}}{dt}\wedge\vec{b}+\vec{a}\wedge\frac{d\vec{b}}{dt}$$

Vettori – integrali

Integrale indefinito $\vec{w} = \int \vec{v}\,dt \quad \Leftrightarrow \quad \vec{v} = \frac{d\vec{w}}{dt}$

Integrale definito $\int_{t_1}^{t_2} \vec{v}\,dt = \vec{w}(t_2) - \vec{w}(t_1)$

Espressioni cartesiane:

$$\frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$\int_{t_1}^{t_2} \vec{v}\,dt = \hat{i}\int_{t_1}^{t_2} v_x dt + \hat{j}\int_{t_1}^{t_2} v_y dt + \hat{k}\int_{t_1}^{t_2} v_z dt$$

Derivate

$Consideriamo\ la\ funzione\ continua\ f(x)\ definita\ nell'intervallo\ [a,b]$

$Si\ definisce\ la\ funzione\ f'(x),\ derivata\ di\ f(x),\ anch'essa\ definita\ nell'intervallo\ [a,b]$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Limite, per Δx che tende a 0, del "rapporto incrementale" della funzione.

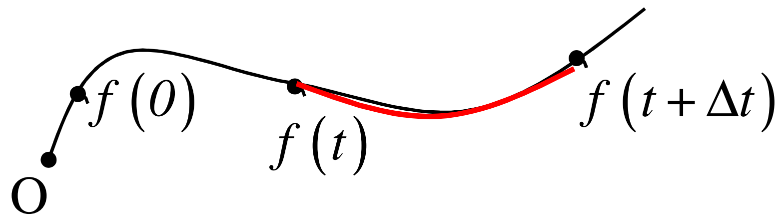
Si scrive anche

$$f'(x) = \frac{df}{dx}$$

Derivate

Caso fisico:
Posizione e velocità di un punto materiale che si muove su una traiettoria data, in funzione del tempo.

$f = f(t)$: *Spazio percorso sulla traiettoria.*



$$v_m = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

La velocità media del punto è il rapporto incrementale della funzione che definisce lo spazio (distanza) percorso sulla traiettoria in funzione del tempo.

Per Δt *che tende a zero,* v_m *tende alla velocità istantanea* $v(t)$.

$$v(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{df}{dt}(t)$$

Derivate

Esempio 1:

$$f(t) = v_c t + s_0$$

$$v_m = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{\cancel{v_c t} + v_c \Delta t + \cancel{s_0} - \cancel{v_c t} - \cancel{s_0}}{\Delta t} = v_c$$

$$v(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = v_c$$

Moto uniforme

Derivate

Esempio 2: $f(t) = kt^2$

$$v_m = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{\cancel{kt^2} + k\Delta t^2 + 2kt\Delta t - \cancel{kt^2}}{\Delta t} = k\Delta t + 2kt$$

$$v(t) = \lim_{\Delta t \to 0}(k\Delta t + 2kt) = 2kt$$

La velocità è il "tasso" di variazione della posizione.

$$a(t) = v'(t) = 2k$$

L'accelerazione è il "tasso" di variazione della velocità.

Moto uniformemente vario

Derivate

Regole di derivazione

Se F(x) è una combinazione di due o più funzioni f(x) , g(x) … :

$$\frac{d}{dx}kf = k\frac{df}{dx} \qquad (k = costante)$$

$$\frac{d}{dx}\left(f \pm g\right) = \frac{df}{dx} \pm \frac{dg}{dx}$$

$$\frac{d}{dx}\left(fg\right) = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(g\frac{df}{dx} - f\frac{dg}{dx}\right)$$

$$Se\ F\left(x\right) = f\left[g\left(x\right)\right] \qquad \frac{dF}{dx} = \frac{df}{dg}\frac{dg}{dx}$$