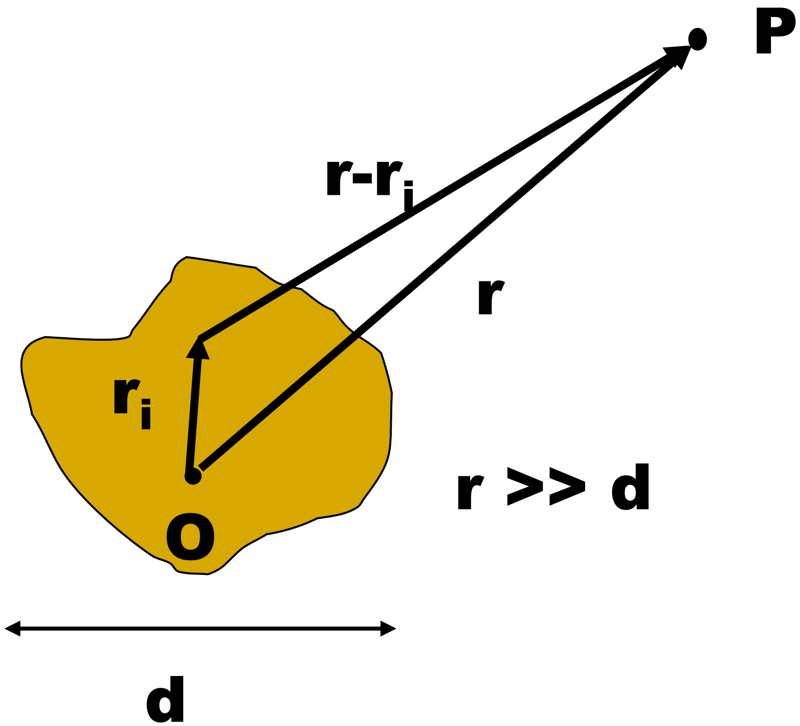
Espansione in multipoli



r >> d

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\vec{r} - \vec{r_i}|}$$

$$f(\vec{r} - \vec{r}_i) \simeq f(\vec{r}) - \vec{r}_i \cdot \vec{\nabla} f(\vec{r})$$

$$\frac{1}{|\vec{r} - \vec{r}_i|} \simeq \frac{1}{r} - \vec{r}_i \cdot \underbrace{\vec{\nabla}\frac{1}{r}}_{-\frac{\vec{e}_r}{r^2}} = \frac{1}{r} + \vec{r}_i \cdot \frac{\vec{e}_r}{r^2} + \cdots$$

Espansione in multipoli

$$V(r)=\frac{1}{4\pi\varepsilon_0}\sum_i q_i\left[\frac{1}{r}+\vec{r}_i\cdot\frac{\vec{e}_r}{r^2}+\cdots\right]=\frac{1}{4\pi\varepsilon_0}\left[\frac{\sum_i q_i}{r}+\frac{\vec{e}_r\cdot\sum_i q_i\vec{r}_i}{r^2}+\cdots\right]$$

$$Q\equiv\sum_i q_i$$

$$\vec{p}\equiv\sum_i q_i\vec{r}_i$$

$$V(r)=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}+\frac{\vec{p}\cdot\vec{e}_r}{r^2}+\cdots\right]$$

K_0 = Q = monopolo
K_1 = dipolo
K_2 = quadrupolo

$$V(r)=\frac{1}{4\pi\epsilon_0}\left(\frac{K_0}{r}+\frac{K_1}{r^2}+\frac{K_2}{r^3}+\ldots\right)$$

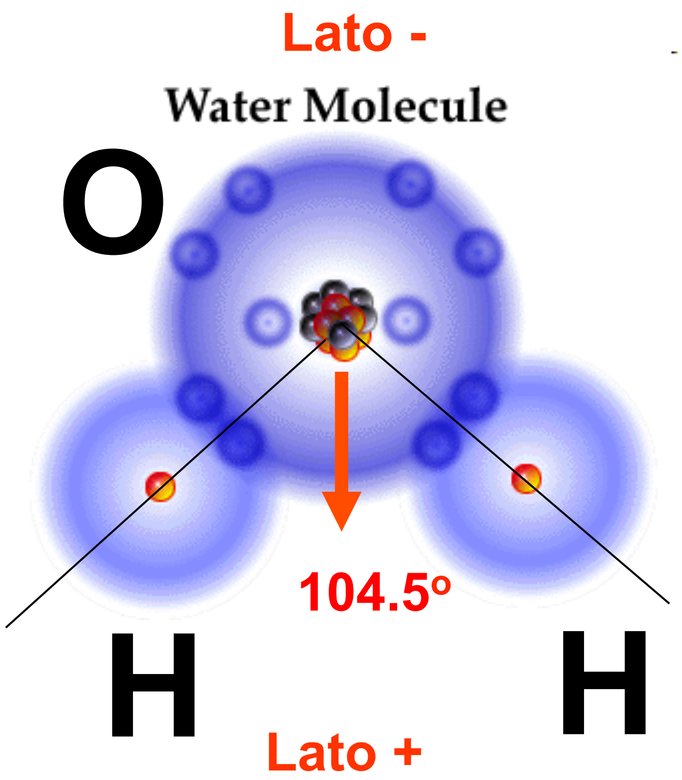
Termine di dipolo

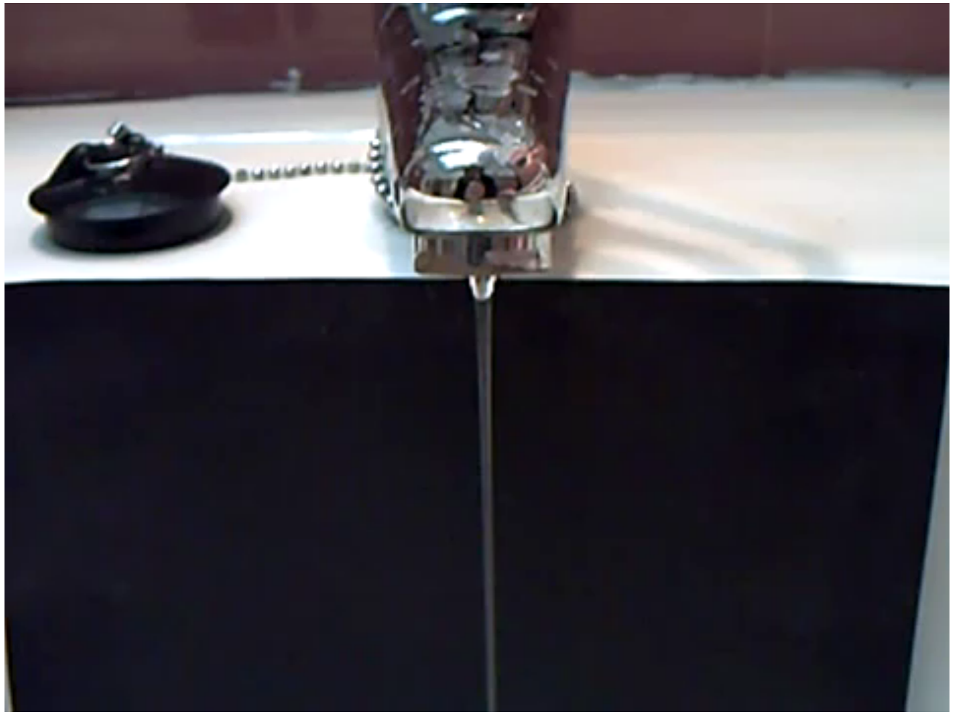
- Se la carica totale e` nulla (Q=0) il momento di dipolo non dipende dalla scelta dell'origine delle coordinate entro la distribuzione di carica (esercizio: dimostrarlo)

- Generalizzazione a distribuzioni continue di carica. Ad esempio:

$$\vec{p} = \int \vec{x}\,'\rho(\vec{x}\,')d^3\vec{x}\,'$$

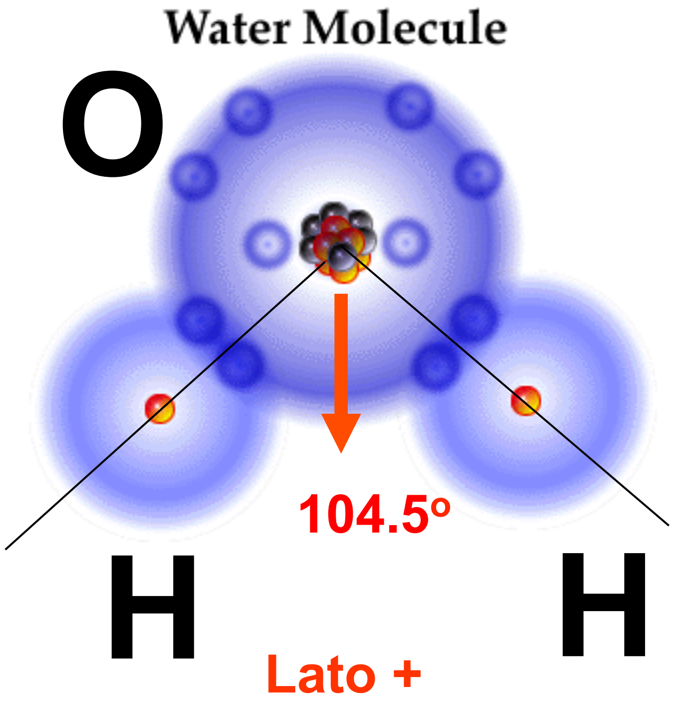
H_2O e` un dipolo



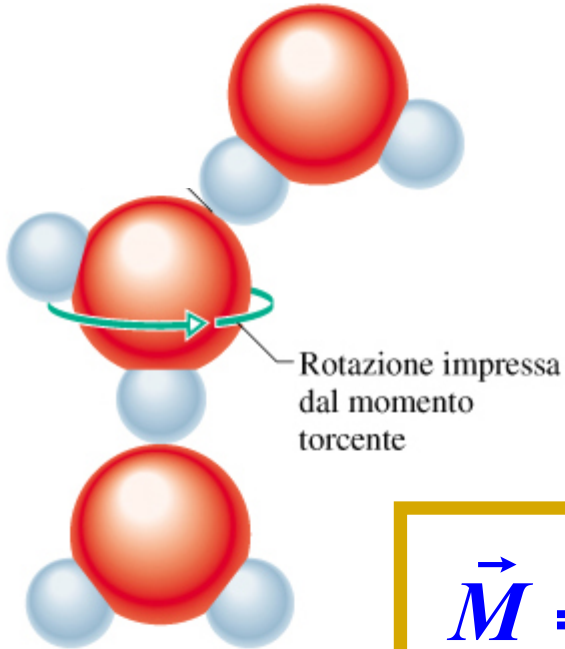


$$p = 6.17 \times 10^{-30}\ \mathrm{C \cdot m}$$

H_2O e` un dipolo

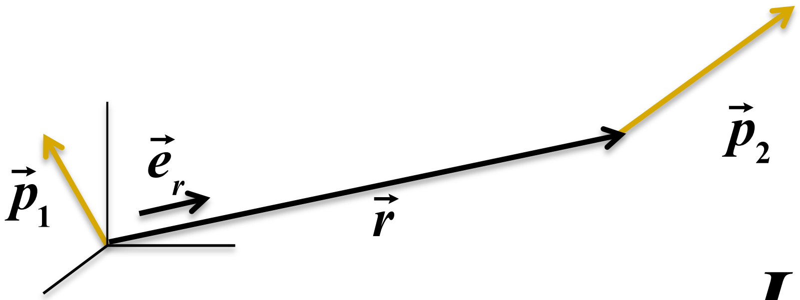


$$p = 6.17 \times 10^{-30}\ \mathrm{C \cdot m}$$



$$\vec{M} = \vec{p} \times \vec{E}$$

Interazione tra due dipoli



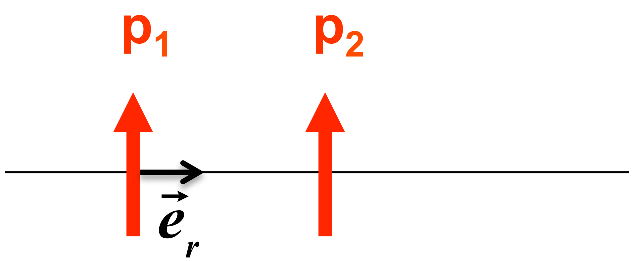
$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\vec{e}_r)\vec{e}_r - \vec{p}\right]$$

$$U = -\vec{p}\cdot\vec{E}$$

$$U_{12} = -\vec{p}_1\cdot\vec{E}_2 = -\vec{p}_2\cdot\vec{E}_1$$

$$U_{12} = \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\vec{e}_r)(\vec{p}_2\cdot\vec{e}_r)\right] = U_{21}$$

$$U_{12} = \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \vec{e}_r)(\vec{p}_2 \cdot \vec{e}_r)\right] = U_{21}$$

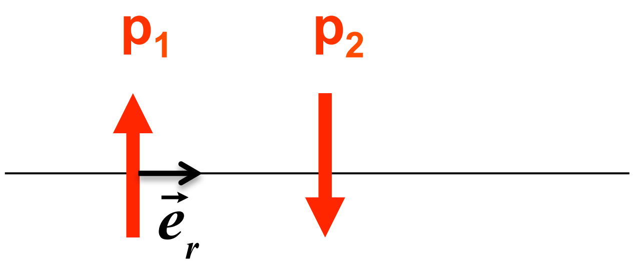


$$U_{12} = \frac{p_1 p_2}{4\pi\varepsilon_0 r^3}$$

forza **repulsiva**

$$\vec{F}_2 = \frac{3\, p_1 p_2}{4\pi\varepsilon_0} \frac{\vec{e}_r}{r^4}$$

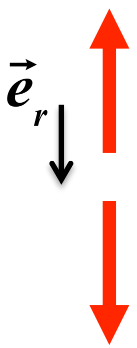
$$U_{12} = \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \vec{e}_r)(\vec{p}_2 \cdot \vec{e}_r)\right] = U_{21}$$



$$U_{12} = -\frac{p_1 p_2}{4\pi\varepsilon_0 r^3}$$

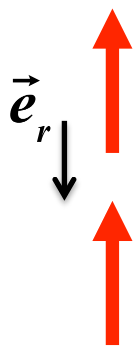
forza **attrattiva** $\vec{F}_2 = -\frac{3\,p_1 p_2}{4\pi\varepsilon_0}\frac{\vec{e}_r}{r^4}$

$$U_{12} = \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \vec{e}_r)(\vec{p}_2 \cdot \vec{e}_r)\right] = U_{21}$$



forza **repulsiva**

$$U_{12} = \frac{2p_1p_2}{4\pi\varepsilon_0 r^3}$$



forza **attrattiva**

$$U_{12} = -\frac{2p_1p_2}{4\pi\varepsilon_0 r^3}$$

Esempio

- Acqua

$$p = 6.2 \cdot 10^{-30} \text{ C m} \qquad r \approx 5 \cdot 10^{-10} \text{ m}$$

$$U_e^{(1)} \approx \frac{p_1 p_2}{4\pi\varepsilon_0 r^3} = \frac{9 \cdot 10^9 (6.2 \cdot 10^{-30})^2}{(5 \cdot 10^{-10})^3} = 2.77 \cdot 10^{-21} J = 17.3 \text{ meV}$$

(1 eV = 1.6 x 10^{-19} J)

$$U_e^{(2)} \approx -2U_e^{(1)} - 34.6 \text{ meV}$$

$$k_B T \approx 26 \text{ meV (300 K)}$$

$$F \sim \frac{U_e}{r} \sim \frac{(3 \div 10) \cdot 10^{-21}}{5 \cdot 10^{-10}} \sim 10^{-11} \text{ N}$$

Esercizio

Data la distribuzione di cariche:

Q_1 1 nC in P_1=(1,1,2) cm

Q_2 5 nC in P_2=(2,2,2) cm

Q_3 -4 nC in P_3=(1,3,2) cm

Q_4 -2 nC in P_4=(3,3,3) cm

calcolare il potenziale elettrostatico nel punto P=(3,4,0) m nell'approssimazione di dipolo, e confrontarlo con il valore esatto.

Termine di quadrupolo

- Esercizio: calcolarlo

suggerimenti

$$f(\vec{r}-\vec{r}_i) = f(\vec{r}) - \vec{r}_i \cdot \vec{\nabla} f(\vec{r}) + \frac{1}{2}\sum_{mn}\left(\frac{\partial^2 f(\vec{r})}{\partial x_n \partial x_m}\right) x_n^{(i)} x_m^{(i)} + \ldots$$

$$\left(f(\vec{r}) = \frac{1}{r}\right)$$

$$\partial_n \partial_m \frac{1}{r} = -\partial_n \frac{x_m}{r^3} = -\frac{\delta_{nm} r^3 - 3r^2 x_m (\frac{x_n}{r})}{r^6} = -\frac{\delta_{nm} - 3\frac{x_n}{r}\frac{x_m}{r}}{r^3}$$

$$\left(\partial_n x_m = \delta_{nm} = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m) \end{cases}\right)$$

$$\left(\partial_n \partial_m \frac{1}{r}\right) x_n^{(i)} x_m^{(i)} = -\frac{\delta_{nm} - 3\frac{x_n}{r}\frac{x_m}{r}}{r^3} x_n^{(i)} x_m^{(i)} = -\frac{|\vec{r}_i|^2 - 3(\vec{r}_i \cdot \vec{e}_r)(\vec{r}_i \cdot \vec{e}_r)}{r^3} = \frac{3(\vec{r}_i \cdot \vec{e}_r)^2 - |\vec{r}_i|^2}{r^3} = \frac{r_i^2 (3\cos^2\theta_i - 1)}{r^3}$$

Termini di monopolo, dipolo e quadrupolo

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{K_0}{r} + \frac{K_1}{r^2} + \frac{K_2}{r^3} + \dots\right)$$

$$K_0 = Q = \sum_i q_i$$

$$K_1 = \vec{p}\cdot\vec{e}_r = \left(\sum_i q_i \vec{r}_i\right)\cdot\vec{e}_r$$

$$K_2 = \sum_i q_i r_i^2 \frac{3\cos^2\theta_i - 1}{2}$$