Recent Developments on the Stochastic Burgers Equation

Michele Baldini, *New York University, Physics Department*
Henry McKean, *New York University, Courant Institute of Mathematical Sciences*

IDEAS:

- Testing ground for the Navier-Stokes turbulence

- Investigation of the invariant measure

Turbulence

The phenomenon of turbulence has been known for centuries and the first mathematical formulation goes back to the works of Navier-Stokes (1820-1840).
It is general folklore that the turbulence is entirely contained in the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mu \nabla^2 \mathbf{v} = \frac{1}{\rho}(\mathbf{f} - \nabla p)$$

PROBLEM: extremely complicated!!! We need simpler models...

The Burgers equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \mu\frac{\partial^2 v}{\partial x^2} = 0$$

where

$v(x,t)$ is the velocity field
μ is the viscosity

Features:

- Same type of non-linearity
- Presence of a dissipative term from which a Reynolds number can be defined
- Many invariances and conservation laws in common

MAIN FLAW

The Burgers equation can be explicitly integrated. Setting

$$v = -2\mu \left(\frac{W_x}{W}\right)$$

it becomes the *Heat equation*

$$\frac{\partial W}{\partial t} = \mu \frac{\partial^2 W}{\partial x^2}$$

Therefore, there is NOT spontaneous arise of randomness by chaotic dynamics.

The STOCHASTIC Burgers equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \mu\frac{\partial^2 v}{\partial x^2} = e \cdot \frac{db}{dt}$$

where

$$e \cdot \frac{db}{dt} = \sum_{n>0} \{\alpha_n \cos 2\pi nx + \beta_n \sin 2\pi nx\} \frac{db_n}{dt}$$

WARNING: this is NOT a partial differential equation in the usual sense (because the Brownian motion is not differentiable). The correct formulation is in terms of a diffusion:

$$dv = e \cdot db - \left(v\frac{\partial v}{\partial x} - \mu\frac{\partial^2 v}{\partial x^2}\right) dt$$

INVARIANT MEASURE

We are interested in the **evolution of the statistical properties** for random solutions. In particular we want to study the asymptotic behavior of the statistic when time goes to infinity.
The limit distribution is called *invariant measure* or *stationary distribution*. If such an object exists, then:

- The distribution of the system converges to the invariant measure no matter what the initial state is.

- If the system is initially distributed according to the invariant measure, then it remains in that state forever.

QUESTIONS:

- Does the stochastic Burgers equation admit an invariant measure?

- If so, how does it look like?

DIFFUSION IN $\mathcal{R}$

A *diffusion* is a Markov process with a continuous path. The general form of a diffusion is:

$$\text{''}\, dX = \sigma(X)db + m(X)dt \,\text{''}$$

$\sigma(X)=$ dispersion

$m(X)=$ drift

$$X_t - X_0 = \int_0^t \sigma(X_s)db_s + \int_0^t m(X_s)ds$$

Pathwise, necessary conditions for the existence of a “nice” (i.e. integrable) invariant measure are:

- The drift must be strong enough to balance the randomness.

- The innovation from the random term $\sigma(X)db$ ought to be killed for large t and the process must stabilize.

This might be not clear at a first sight...

The ORNSTEIN-UHLENBECK process

It was introduced to describe the *velocity* field of a Brownian pollen grain on the surface of a liquid. In fact, if $X(t)$ was the position, the velocity would be meaningless.

$$dX = db - X dt$$

The solution is

$$X_t = e^{-t} X_0 + \int_0^t e^{s-t} db_s$$

Note:

- The process rapidly forgets where it came from

- In the integral term, the innovation at large time enters with amplitude 1

- The limit for $t \to \infty$ is not defined

Show that the process stabilize

No HARM in changing the Brownian motion

Fix a time T and substitute:

$$b(t) \longrightarrow b^{\downarrow}(t) \equiv b(T) - b(T-t)$$

$$dX^{\downarrow} = db^{\downarrow} - X^{\downarrow} dt$$

The solution is

$$\begin{aligned} X_t^{\downarrow} &= e^{-t} X_0 + \int_0^t e^{s-t} db_s^{\downarrow} \\ &= e^{-t} X_0 + \int_0^t e^{-s} db_s \end{aligned}$$

Note: <u>the innovation is damped exponentially fast!!</u>

CONCLUSIONS

Basic hope: every diffusion with a "nice" invariant measure features this mechanism. Unfortunately, computation becomes horrible but at least it will keep me busy for the next year...