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RECENT DEVELOPMENTS ON THE STOCHASTIC BURGERS EQUATION

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It is well known among physicists and mathematicians interested in fluid dynamics that the Burgers equation $v_t + vv_x - \mu v_{xx} = 0$ ([B], [LV]) is NOT a good model for the phenomenon of turbulence [F], even though it displays a simultaneous presence of nonlinear as well as dissipative terms, essential features for developing a chaotic behaviour. The main reason for this inadequacy is that the equation doesn't contain the pressure, which, thanks to its non-local action, is deemed to be the responsible term that introduces the statistic.

A different and more propitious situation occurs when we add "by hands" a stochastic force on the right hand side of the equation, the simplest being a superposition of independent white noises multiplied by a periodic function, namely

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \mu\frac{\partial^2 v}{\partial x^2} = e \cdot \frac{db}{dt}$$

where

$$e \cdot \frac{db}{dt} = \sum_{n>0} \{\alpha_n \cos 2\pi nx + \beta_n \sin 2\pi nx\} \frac{db_n}{dt}$$

This, of course, has to be interpreted as a stochastic differential equation and therefore Ito's stochastic integration theory is needed [MK]. In particular we are interested in the statistical properties of the solution ([EKMS], [S], [P]) and in the study of the invariant measure [DPZ], one of the main objects in ergodic theory since it gives a representation of a statistical steady state in which the equation settles down. Here we present briefly two of the ideas involved in this investigation:

1) The main idea in the proof of an existence of an invariant measure goes back to W. Doeblin, one of the pioneers of probability theory and great investigator of Markov chains. The briefness of his approach makes it very elegant and manageable [MK2].

2) In order to study asymptotic behaviour of the process we employ a "substitution of the randomness". Roughly speaking, a stochastic process $t \to X(t) \in \mathcal{R}^n$ is a probability measure defined on the infinite dimensional space of paths. An explicit representation of this measure is a difficult task, even in the "simplest" case of the Wiener measure. We explore the shape of this invariant measure replacing the original Brownian motion with a different Brownian motion which runs backward in time. This operation does not spoil the statistic of the process but luckily brings the innovation back to the origin and helps the path to stay confined.

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